NONASYMPTOTIC FORMULAE FOR THE DISTRIBUTION OF THE MAXIMUM OF SMOOTH GAUSSIAN PROCESSES

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Nonasymptotic Formulae for the Distribution of the Maximum of Smooth Gaussian Processes

Jean Diebolt and Christian Posse

Abstract

We derive an integral formula for the density of the maximum of smooth Gaussian processes. This expression induces explicit lower and upper bounds which are in general asymptotic to the density. Our constructive approach relies on a geometric representation of Gaussian processes involving a unit speed parameterized curve embedded in the unit sphere. [1]

1 Introduction

Let $X(t), t \in I = [0,T]$ be a real Gaussian process with mean zero and continuous sample functions. Numerous papers have been devoted to the study of

$$Z = \sup_{t \in I} X(t)$$

(see, e.g., Adler, 1981, 1990, Leadbetter *et al.*, 1983, Samorodnitsky, 1991, for a survey). It turns out that the exact distribution of Z is known only for the Wiener process, the Brownian Bridge $B(t)$, $B(t) - \int_0^1 B(u)du$ (Darling, 1983), the Ornstein-Uhlenbeck process, the integrated Wiener process (see, e.g., Lachal, 1991), a class of "sawtooth" processes (see, e.g., Cressie, 1980) and the random cosine wave $X(t) = \xi_1 \cos(t) + \xi_2 \sin(t)$, where ξ_1 and ξ_2 are independent, identically distributed $\mathcal{N}(0,1)$. Otherwise, the results obtained are either asymptotic formulae for the tail probabilities $P\{Z > a\}$ as $a \to \infty$ or nonasymptotic upper bounds for $P\{Z > a\}$.

[1] AMS 1991 subject classifications. Primary 60G15, 60G70. Secondary 60G17.
Key words and phrases. Differential geometry, Gaussian processes, extreme value, nonasymptotic formulae, density.

In this paper, we consider the class of centered Gaussian processes of the form

$$X(t) = \frac{1}{\tau_X(t)} \sum_{j=1}^{n} \xi_j g_j(t), \quad 2 \leq n \leq \infty, \tag{1}$$

where ξ_j, $j \geq 1$, are independent, identically distributed $\mathcal{N}(0,1)$, $\sum_{j=1}^{n} g_j^2(t) \equiv 1$, and the functions $\tau_X(t) > 0$ and $g_j(t)$, $j \geq 1$, are sufficiently smooth.

The existence of the Karhunen-Loève expansion for Gaussian processes with continuous sample functions ensures that this class is very large (Adler, 1990). First, all Gaussian processes with smooth covariance function have such a representation. Second, more general processes can be arbitrarily approximated by processes of this class with respect to the uniform norm. In addition, in the context of stationary Gaussian processes, the discretization of the harmonic representation $X(t) = \int_{-\infty}^{\infty} \exp(it\lambda) d\Lambda(\lambda)$ (where Λ is a Gaussian measure with independent increments) or of the filtered white noise representation $X(t) = \int_{-\infty}^{\infty} h(t-s) dW(s)$ (where $W(t)$ is the Wiener process) induces the form (1). See also Berman (1988) for stochastic modelizations leading to processes (1).

Berman (1988) studies the asymptotic behavior of the tail probabilities of the supremum Z of processes of the form (1) with $\tau_X(t) \equiv 1$, finite n and orthogonally invariant joint distribution of $(\xi_1, \ldots, \xi_n)$. In the normal case, his results become (Theorem 18.1, p. 37)

$$P\{Z > a\} \sim \frac{L}{2\pi} \exp(-a^2/2) \quad (a \to \infty), \tag{2}$$

where $L = \int_0^T (\sum_{j=1}^{n} \dot{g}_j^2(t))^{1/2} \, dt$, and (Corollary 17.1, p. 36)

$$P\{Z > a\} \leq \frac{L}{2\pi} \exp(-a^2/2) + \int_a^{\infty} (2\pi)^{-1/2} \exp(-x^2/2) \, dx \quad (a > 0). \tag{3}$$

Johnstone and Siegmund (1989) consider processes of the form (1) with $\tau_X(t) \equiv 1$, finite n and $(\xi_1, \ldots, \xi_n)$ uniformly distributed on the unit sphere. By making use of the well-known connection between the standard Gaussian distribution in $\mathbb{R}^n$ and the uniform distribution on the unit sphere of $\mathbb{R}^n$, we can adapt their result (Theorem 3.3, p. 190) to our context. It turns out that the resulting upper bound is the expression (3) and then is independent of n !

Sun (1993) investigates an asymptotic expansion for the tail probabilities of the maximum of smooth Gaussian random fields with unit variance. In the special case of processes, her results concern periodic processes of the form (1) with $\tau_X(t) \equiv 1$. For finite n, Sun obtains the asymptotic formula (2) (Theorem 3.1, p. 40) as a consequence of Weyl's Formula for the volume of tubes around a manifold embedded in the unit sphere. For infinite n, (2) still holds under additional assumptions, otherwise it becomes an upper bound (Theorems 3.2 and 3.3, p. 41).

Our approach is based on the interpretation of the functions $g_j(t)$, $j \geq 1$, as a parameterization of a curve embedded in the unit sphere of $\mathbb{R}^n$ or of the space

of square summable sequences. With the canonical moving frame induced by this parameterization, we describe each level manifold $\{z = b\}$, $b \in \mathbb{R}$, of the functional

$$z = \sup_{t \in I} \frac{1}{\tau_X(t)} \sum_{j=1}^{n} x_j g_j(t),$$

where $\{x_j : j \geq 1\}$ is a realization of $\{\xi_j : j \geq 1\}$, as an envelope of the family of hyperplanes $\{\tau_X(t)^{-1} \sum_{j=1}^{n} x_j g_j(t) = b : t \in I\}$. This technique enables us to express the density $f_Z(b)$ of Z as the canonical volume of $\{z = b\}$, leading to an integral formula for $f_Z(b)$. This method provides accurate lower and upper bounds for $f_Z(b)$ and $P\{Z > a\}$ (see Diebolt and Posse, 1994, for numerical investigations). Moreover, it allows us to handle processes with varying variance, to deal with the case $n = \infty$ and to manage the boundary effects with great care.

The remainder of the paper is organized as follows. Our main results are presented in Section 2 for $n < \infty$ and are extended to the infinite case in Section 3. The theorems in Sections 2 and 3 are proved in Section 4. Some related known results of differential geometry are briefly introduced in the Appendix.

Notation. Throughout the paper, a.e. means almost every, ℓ^2 refers to the Hilbert space of square summable sequences, $\mathbf{x} = (x_1, x_2, \ldots)$ is an element either of $\mathbb{R}^n$ or ℓ^2, $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=1}^{n} x_j y_j$, $2 \leq n \leq \infty$, $\|\mathbf{x}\| = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}$, $Vect(\mathbf{x}_1, \ldots, \mathbf{x}_d)$ and $Vect^{\perp}(\mathbf{x}_1, \ldots, \mathbf{x}_d)$ denote the linear subspace spanned by $(\mathbf{x}_1, \ldots, \mathbf{x}_d)$ and its orthogonal respectively, $\mathrm{Gram}(\mathbf{x}_1, \ldots, \mathbf{x}_d)$ is the determinant of the matrix G with entries $G_{ij} = \langle \mathbf{a}_i, \mathbf{a}_j \rangle$ (note that $\det G = \det^2(\mathbf{x}_1, \ldots, \mathbf{x}_d)$). μ_n is the Gaussian measure on $\mathbb{R}^n$ with density $\varphi_n(\mathbf{x}) = (2\pi)^{-n/2} \exp(-\|\mathbf{x}\|^2/2)$, $\varphi(x) = \varphi_1(x)$ and $\Phi(x) = \int_{-\infty}^{x} \varphi(y) dy$. By convention, $\varphi(\infty) = 0$ and $\Phi(\infty) = 1$. $C^m(A)$ denotes the set of functions $A \to \mathbb{R}$ having kth-order continuous derivatives for $k = 1, \ldots, m$. The partial derivatives $\partial^{k+l} r(x, y)/\partial x^k \partial y^l$ where $r \in C^{k+l}(A)$ is written $D_{kl} r(x, y)$. The Jacobian matrix of a differentiable mapping $\mathbf{p} : \mathbb{R}^n \to \mathbb{R}^n$ is denoted $D\mathbf{p}$.

2 Main results

Let $X(t), t \in I = [0, T]$, be a Gaussian process with mean 0 and variance $\sigma_X^2(t) > 0$, of the form

$$X(t) = \frac{1}{\tau_X(t)} \langle \Xi, \mathbf{g}(t) \rangle, \tag{4}$$

where $\tau_X(t) = \sigma_X^{-1}(t)$, $\mathbf{g}(t) = (g_1(t), \ldots, g_n(t))$, $n \geq 2$, and $\Xi = (\xi_1, \ldots, \xi_n)$ is a Gaussian r.v. with zero mean and identity covariance matrix. With the representation (4), the covariance function $r_X(t_1, t_2)$ of $X(t)$ is given by

$$r_X(t_1, t_2) = \frac{1}{\tau_X(t_1)\tau_X(t_2)} \langle \mathbf{g}(t_1), \mathbf{g}(t_2) \rangle.$$

Since $\sigma_X^2(t) = r_X(t, t) = \|\mathbf{g}(t)\|^2/\tau_X^2(t)$, $\|\mathbf{g}(t)\| \equiv 1$ and $\mathbf{g}(t)$ parameterizes a curve γ embedded in the unit sphere S^{n-1} in $\mathbb{R}^n$.

In the following, we assume that

(C1) the functions $\tau_X(t), g_1(t), \ldots, g_n(t)$ are in $\mathcal{C}^2(I)$;

(C2) $\|\mathbf{g}'(t)\| \neq 0$ for all $t \in I$;

(C3) $\{t : c_g(t) = 0\} \subset \{t : \tau_X(t) - \tau_X'(t)\langle \mathbf{g}'(t), \mathbf{g}''(t)\rangle / \|\mathbf{g}'(t)\|^3 + \tau_X''(t)/\|\mathbf{g}'(t)\|^2 > 0\}$ where the smooth function $c_g(t) \geq 0$ defines the geodesic curvature of γ at the point $\mathbf{g}(t)$ (see Appendix);

(C4) whenever $\mathbf{g}(t)$, $\mathbf{g}'(t)$ and $\mathbf{g}(t')$ are linearly dependent for $t' \neq t$, $\tilde{\mathbf{g}}(t') \neq \tilde{\mathbf{g}}(t)$ where $\tilde{\mathbf{g}}(t) = \tau_X(t)\mathbf{g}(t) + \tau_X'(t)\mathbf{g}'(t)/\|\mathbf{g}'(t)\|^2$.

Remark. We can relax condition **(C2)** allowing a finite set I_0 of points such that $\|\mathbf{g}'(t)\|^2 = 0$ for $t \in I_0$. **(C2)** avoids additional technicalities in the proofs of our results.

Remark. If $\sigma_X(t) \equiv 1/\tau$ is constant, **(C3)** is automatically satisfied, $\tilde{\mathbf{g}}(t) = \tau\mathbf{g}(t)$ and **(C4)** means that the curve γ has no self-intersection, i.e. $t' \neq t \Rightarrow \mathbf{g}(t') \neq \mathbf{g}(t)$ for all $t, t' \in (0, T)$ if $\mathbf{g}(t)$ is T-periodic and for all $t, t' \in [0, T]$ otherwise.

We are interested in finding estimates for the distribution of

$$Z = \sup_{t \in I} X(t).$$

The key idea of our approach is to transform this problem into a geometric problem concerning the standard Gaussian measure of some convex subsets of $\mathbb{R}^n$. Our main result is an integral formula for the density f_Z of Z which is stated in Theorem 1. The derivation of this formula which is sketched below is greatly simplified if we parameterize γ with unit speed. This can be done without loss of generality when condition **(C2)** holds. Let us define the Gaussian process $Y(s)$, $s \in J$, as

$$Y(s) = \frac{1}{\tau_Y(s)} \langle \Xi, \mathbf{f}(s) \rangle,$$

where $\tau_Y(s) = \tau_X(\lambda^{-1}(s))$, $\mathbf{f}(s) = \mathbf{g}(\lambda^{-1}(s))$ and $s = \lambda(t) = \int_0^t \|\mathbf{g}'(u)\|\, du$ defines a unit speed parameterization of γ, $J = [0, L]$ with $L = |\gamma| = \lambda(T)$. Then, we have

$$Z = \sup_{t \in I} X(t) = \sup_{s \in J} Y(s).$$

The covariance function of $Y(s)$ is given by $r_Y(s_1, s_2) = r_X(\lambda^{-1}(s_1), \lambda^{-1}(s_2))$. Moreover, $\|\mathbf{f}(s)\| = \|\mathbf{f}'(s)\| \equiv 1$ for all $s \in J$. Note that in terms of $s = \lambda(t)$, **(C3)** becomes: $\{s : c_g(s) = 0\} \subset \{s : \tau_Y(s) + \tau_Y''(s) > 0\}$.

For simplicity, all our results will be expressed in terms of this unit speed parameterization. However, simple transformations give the corresponding formulae expressed in the original parameterization. In practice, only the latter are used.

Our method relies on the existence of an orthonormal moving frame ($\mathbf{f}(s)$, $\mathbf{T}(s), \mathbf{K}_1(s), \ldots, \mathbf{K}_{n-2}(s)$) of $\mathbb{R}^n$ such that the space tangent to S^{n-1} at $\mathbf{f}(s)$ is spanned by $(\mathbf{T}(s), \mathbf{K}_1(s), \ldots, \mathbf{K}_{n-2}(s))$. See the Appendix for more details.

Let us consider the realization of $(\Omega, \mathcal{A}, P)$ as $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mu_n)$ where $\mathcal{B}(\mathbb{R}^n)$ denotes the Borel σ-field of $\mathbb{R}^n$. It follows that

$$Y(s, \mathbf{x}) = \frac{1}{\tau_Y(s)} \langle \mathbf{x}, \mathbf{f}(s) \rangle, \qquad \mathbf{x} \in \mathbb{R}^n,$$

is a realization of the process $Y(s)$ and

$$P\{Z \leq a\} = \mu_n(C_a), \qquad a \in \mathbb{R}, \tag{5}$$

where

$$C_a = \{\mathbf{x} \in \mathbb{R}^n : \sup_{s \in J} Y(s, \mathbf{x}) \leq a\}.$$

The boundaries ∂C_b of C_b, $b \leq a$, partition C_a. Indeed, by Lemma 6 in Section 4,

$$\partial C_b = \{\mathbf{x} \in \mathbb{R}^n : \sup_{s \in J} Y(s, \mathbf{x}) = b\}.$$

This suggests that a suitable change of variable will express $\mu_n(C_a)$ as an integral over $b \leq a$ of appropriate superficial measures of ∂C_b:

$$\mu_n(C_a) = \int_{-\infty}^{a} \psi_b(\partial C_b)\, db = \int_{-\infty}^{a} f_Z(b)\, db. \tag{6}$$

Such a decomposition can be worked out basically—for simplicity, we assume γ closed here, the case γ not closed can be treated essentially in the same way—because it is possible (see Lemma 8) to parameterize ∂C_b by

$$\mathbf{p}_b(s, u) = c_1(b, s)\mathbf{f}(s) + c_2(b, s)\mathbf{T}(s) + \sum_{j=1}^{n-2} u_j \mathbf{K}_j(s),$$

where $s \in J$, $u = (u_1, \ldots, u_{n-2}) \in D_b(s)$, $c_1(b, s)$ and $c_2(b, s)$ are defined in terms of b, $\tau_Y(s)$ and its derivative, and $D_b(s)$ is a closed convex subset of $\mathbb{R}^{n-2}$. We show in Lemmas 10 and 11 that the transformation $\mathbf{p} : (b, s, u) \to \mathbf{p}_b(s, u)$ is a C^1-diffeomorphism from an open subset of $\mathbb{R}^n$ into $\mathbb{R}^n$. By the change-of-variable formula and Fubini's Theorem, we have, for all $A \in \mathcal{B}(\mathbb{R}^n)$,

$$\begin{aligned}
\mu_n(A) &= \int_{(b,s,u) \in \mathbf{p}^{-1}(A)} \varphi_n(\mathbf{p}(b, s, u)) \mathrm{Gram}^{1/2} D\mathbf{p}(b, s, u)\, db\, ds\, du \\
&= \int_{b \in \mathbb{R}} \int_{(s,u) \in \mathbf{p}_b^{-1}(A \cap \partial C_b)} \varphi_n(\mathbf{p}_b(s, u)) \mathrm{Gram}^{1/2} D\mathbf{p}(b, s, u)\, ds\, du\, db \\
&= \int_{b \in \mathbb{R}} \psi_b(A \cap \partial C_b)\, db.
\end{aligned}$$

Since $C_a \cap \partial C_b = \partial C_b$ if $b \leq a$ and $C_a \cap \partial C_b = \emptyset$ otherwise, we obtain (6).

Remark. This geometric approach has a probabilistic interpretation in terms of conditional distributions. Since $\mu_n(A) = P\{\Xi \in A\}$,

$$\mu_n(A) = \int_{b\in\mathbb{R}} \frac{\psi_b(A\cap\partial C_b)}{\psi_b(\partial C_b)} f_Z(b)\,db \quad \text{and} \quad P\{\Xi\in A\} = \int_{b\in\mathbb{R}} P\{\Xi\in A \mid Z=b\} f_Z(b)\,db$$

for all $A \in \mathcal{B}(\mathbb{R}^n)$, it results that $\psi_b(A\cap\partial C_b)/\psi_b(\partial C_b)$ is a regular version of the conditional probability $P\{\Xi \in A \mid Z = b\}$. Similarly, by Bayes formula, the distribution of Z conditioned on $\{\Xi \in A\}$ has density $\psi_b(A\cap\partial C_b)/\mu_n(A)$.

Remark. The canonical superficial measure on ∂C_b induced by $\varphi_n(\mathbf{x})$ is defined by

$$\psi_b^*(A\cap\partial C_b) = \int_{(s,u)\,\in\,\mathbf{p}_b^{-1}(A\cap\partial C_b)} \varphi_n(\mathbf{p}_b(s,u))\mathrm{Gram}^{1/2}D\mathbf{p}_b(s,u)\,ds\,du$$

(Berger and Gostiaux, 1988, p. 203). If $\sigma_Y(s) \equiv 1/\tau$ is constant, it results from Lemma 10 that $\psi_b(A\cap\partial C_b) = \tau\psi_b^*(A\cap\partial C_b)$ for all $A \in \mathcal{B}(\mathbb{R}^n)$. Otherwise, ψ_b appears as a weighted version of ψ_b^*, with weight $\tau_Y(s)$:

$$\psi_b(A\cap\partial C_b) = \int_{(s,u)\in\mathbf{p}_b^{-1}(A\cap\partial C_b)} \varphi_n(\mathbf{p}_b(s,u))\tau_Y(s)\mathrm{Gram}^{1/2}D\mathbf{p}_b(s,u)\,ds\,du. \quad (7)$$

The above approach leads to the following expression for $f_Z(b)$:

Theorem 1 *Under conditions* **(C1)** $-$ **(C4)**, *the density of Z is given by*

$$f_Z(b) =$$

$$\int_0^L \int_{D_b(s)} \tau_Y(s)(b(\tau_Y(s)+\tau_Y''(s)) - u_1 c_g(s))\varphi_n(\mathbf{p}(b,s,u))\,du_1\ldots du_{n-2}\,ds + \delta_Z(b), \quad (8)$$

where

$$\delta_Z(b) = \begin{cases} 0 & \gamma \textit{ closed}, \\ \tau_Y(0)\varphi(b\tau_Y(0))\mu_{n-1}(G_b(0)) + \tau_Y(L)\varphi(b\tau_Y(L))\mu_{n-1}(G_b(L)) & \textit{otherwise}, \end{cases}$$

with

$$D_b(s) = \{u = (u_1,\ldots,u_{n-2}) \in \mathbb{R}^{n-2} : \sup_{s'\in J} \frac{1}{\tau_Y(s')}\langle \mathbf{p}(b,s,u), \mathbf{f}(s')\rangle \le b\},$$

$$\mathbf{p}(b,s,u) = b(\tau_Y(s)\mathbf{f}(s) + \tau_Y'(s)\mathbf{T}(s)) + \sum_{j=1}^{n-2} u_j\mathbf{K}_j(s),$$

$$G_b(l) = \{v = (v_1,\ldots,v_{n-1}) \in \mathbb{R}^{n-1} : \sup_{s'\in J} \frac{1}{\tau_Y(s')}\langle \mathbf{p}_{b,l}(v), \mathbf{f}(s')\rangle \le b\},$$

and

$$\mathbf{p}_{b,l}(v) = b\tau_Y(l)\mathbf{f}(l) + v_{n-1}\mathbf{T}(l) + \sum_{j=1}^{n-2} v_j\mathbf{K}_j(l) \qquad l = 0, L.$$

Remark. Taylor expansions of order 1 of $\langle \mathbf{p}_{b,0}(v), \mathbf{f}(s)\rangle - b\tau_Y(s)$ (respectively $\langle \mathbf{p}_{b,L}(v), \mathbf{f}(s)\rangle - b\tau_Y(s)$) around l, $l = 0, L$, show that $G_b(0)$ (respectively $G_b(L)$) has Lebesgue measure zero in $\mathbb{R}^{n-1}$ if γ is closed.

Corollary 2 (Upper bound) *Under conditions* **(C1)** $-$ **(C4)**,

$$f_Z(b) \le M(b) =$$

$$\frac{b}{2\pi}\int_0^L \tau_Y(s)(\tau_Y(s)+\tau_Y''(s)) \exp\left(-\frac{b^2}{2}(\tau_Y^2(s)+\tau_Y'^2(s))\right) \Phi\left(\frac{b(\tau_Y(s)+\tau_Y''(s))}{c_g(s)}\right) ds$$

$$+\frac{1}{2\pi}\int_0^L \tau_Y(s)c_g(s) \exp\left(-\frac{b^2}{2}(\tau_Y^2(s)+\tau_Y'^2(s))\right) \varphi\left(\frac{b(\tau_Y(s)+\tau_Y''(s))}{c_g(s)}\right) ds + \delta_M(b) \tag{9}$$

where

$$\delta_M(b) = \begin{cases} 0 & \gamma \textit{ closed,} \\ \tau_Y(0)\varphi(b\tau_Y(0))\Phi(b\tau_Y'(0)) + \tau_Y(L)\varphi(b\tau_Y(L))(1-\Phi(b\tau_Y'(L))) & \textit{otherwise.} \end{cases}$$

Remark. The corresponding expressions in terms of $t = \lambda^{-1}(s)$ for $M(b)$ and $\delta_M(b)$ are obtained by replacing L by T, ds by $\|\mathbf{g}'(t)\|dt$, $c_g(s)$ by $c_g(t)$ (see Appendix), $\tau_Y(s)$ by $\tau_X(t)$, $\tau_Y'(s)$ by $\tau_X'(t)/\|\mathbf{g}'(t)\|$ and $\tau_Y''(s)$ by $-\tau_X'(t)\langle \mathbf{g}'(t), \mathbf{g}''(t)\rangle/\|\mathbf{g}'(t)\|^3 + \tau_X''(t)/\|\mathbf{g}'(t)\|^2$.

Corollary 2 is a direct consequence of Lemma 9 in Section 4 and can be used to derive a lower bound for $f_Z(b)$, $b > 0$. Indeed, the integral formula (8) can be rewritten as

$$f_Z(b) = \frac{b}{2\pi}\int_0^L \tau_Y(s)(\tau_Y(s)+\tau_Y''(s)) \exp\left(-\frac{b^2}{2}(\tau_Y^2(s)+\tau_Y'^2(s))\right) \mu_{n-2}(D_b(s))\, ds$$

$$-\frac{1}{2\pi}\int_0^L \tau_Y(s)c_g(s)\exp\left(-\frac{b^2}{2}(\tau_Y^2(s)+\tau_Y'^2(s))\right)\int_{D_b(s)} u_1\varphi_{n-2}(u)\, du_1 \ldots du_{n-2}\, ds + \delta_Z(b). \tag{10}$$

By definition of $D_b(s)$ and relation (5), $\mu_{n-2}(D_b(s))$ and $\int_{D_b(s)} u_1\varphi_{n-2}(u)\,du$ can be interpreted in terms of $P(\sup_{s'\in J} W_s(s') \le b)$, where $W_s(s')$ is a suitable Gaussian process of the form (4) for a.e. $s \in J$. Therefore, we can make use of Theorem 1 and Corollary 2 to provide upper bounds for $P(\sup_{s'\in J} W_s(s') > a)$ and the absolute value of the second term in the right-hand side of (10). This approach requires that the process $W_s(s')$ satisfies conditions **(C1)**–**(C4)**. The resulting assumptions are still mild but very technical. For simplicity, we give sufficient conditions for $n \ge 5$:

(C5) the function $\tau_X(t)$ is in $C^3(I)$ and the functions $g_1(t), \ldots, g_n(t)$ are in $C^6(I)$;

(C6) for all $t \in I$, $\tau_X(t) - \tau'_X(t)\langle \mathbf{g}'(t), \mathbf{g}''(t)\rangle / \|\mathbf{g}'(t)\|^3 + \tau''_X(t)/\|\mathbf{g}'(t)\|^2 \neq 0$;

(C7) for all $t \in I$, $\beta_t(t') = \tau_X(t') - \tau_X(t)\langle \mathbf{g}(t), \mathbf{g}(t')\rangle - \tau'_X(t)\langle \mathbf{g}'(t), \mathbf{g}(t')\rangle / \|\mathbf{g}'(t)\|^2 > 0$ for all $t' \neq t$, $t' \in I$;

(C8) for a.e. $t \in I$, the vectors $\mathbf{g}(t)$, $\mathbf{g}'(t)$, $\mathbf{g}(t')$, $\mathbf{g}'(t')$ and $\mathbf{g}''(t')$ are linearly independent for all $t' \neq t$, $t' \in I$;

(C9) for a.e. $t \in I$, the vectors $\mathbf{g}(t)$, $\mathbf{g}'(t)$, $\mathbf{g}(t')$, $\mathbf{g}'(t')$ and $\mathbf{g}(t'')$ are linearly independent for all $t' \neq t$, $t'' \neq t$, $t'' \neq t'$ and $t', t'' \in I$.

Remark. In terms of $s = \lambda(t)$, (**C6**) and (**C7**) say that for all $s \in J$, $\tau_Y(s) + \tau''_Y(s) \neq 0$ and $\beta_s(s') = \tau_Y(s') - \tau_Y(s)\langle \mathbf{f}(s), \mathbf{f}(s')\rangle - \tau'_Y(s)\langle \mathbf{T}(s), \mathbf{f}(s')\rangle > 0$ for all $s' \neq s$, $s' \in J$. (**C7**) implies that $\tau_Y(s) + \tau''_Y(s) \geq 0$ for all $s \in J$ since $\beta_s(s+h) = (\tau_Y(s) + \tau''_Y(s))h^2/2 + o(h^2)$ as $h \to 0$.

Remark. Under (**C8**), $\|\mathbf{g}'(t)\| \neq 0$ and $c_g(t) > 0$ for all $t \in I$, and (**C2**) and (**C3**) applied to the process $W_s(s')$ hold for a.e. $s \in J$. Under (**C9**), (**C4**) applied to $X(t)$ and $W_s(s')$ for a.e. $s \in J$ holds. (**C7**) implies (**C4**). If $\sigma_X(t) \equiv 1/\tau$ is constant, the reverse implication is true.

More precisely, let us define the functions

$$\begin{aligned}
\alpha_s(s') &= (1 - \langle \mathbf{f}(s), \mathbf{f}(s')\rangle^2 - \langle \mathbf{T}(s), \mathbf{f}(s')\rangle^2)^{1/2}, && (11)\\
\tau_s(s') &= \beta_s(s')/\alpha_s(s'), && (12)\\
\eta(s) &= \inf_{s' \in J} \tau_s(s'), && (13)\\
r_s(s'_1, s'_2) &= \alpha_s^{-1}(s'_1)\alpha_s^{-1}(s'_2) \\
&\quad \times (\langle \mathbf{f}(s'_1), \mathbf{f}(s'_2)\rangle - \langle \mathbf{f}(s), \mathbf{f}(s'_1)\rangle\langle \mathbf{f}(s), \mathbf{f}(s'_2)\rangle - \langle \mathbf{T}(s), \mathbf{f}(s'_1)\rangle\langle \mathbf{T}(s), \mathbf{f}(s'_2)\rangle) && (14)
\end{aligned}$$

if $s'_1 \neq s$ and $s'_2 \neq s$, $r_s(s, s'_2) = \langle \mathbf{f}(s) + \mathbf{f}''(s), \mathbf{f}(s'_2)\rangle / (c_g(s)\alpha_s(s'_2))$ and $r_s(s,s) = 1$.

By condition (**C9**), $\alpha_s(s') = (\sum_{j=1}^{n-2}\langle \mathbf{K}_j(s), \mathbf{f}(s')\rangle^2)^{1/2} > 0$ for all $s' \neq s$, $s, s' \in J$. By condition (**C5**), $r_s(s'_1, s'_2)$ has continuous partial derivatives $D_{kl}r_s(s'_1, s'_2)$ for $0 \leq k, l \leq 4$. Moreover, $\tau_s(s') > 0$ and $\eta(s) > 0$ for all $s, s' \in J$.

For a.e. $s \in J$, $W_s(s') = \tau_s^{-1}(s')\langle \mathbf{\Omega}, \mathbf{k}_s(s')\rangle$, where $\mathbf{\Omega} = (\omega_1, \ldots, \omega_{n-2})$ is a Gaussian r.v. with mean zero and identity covariance matrix and $\mathbf{k}_s(s') = (k_{s,1}(s'), \ldots, k_{s,n-2}(s'))$ parameterizes a curve γ_s on the unit sphere of $\mathbb{R}^{n-2}$. Moreover, $r_s(s'_1, s'_2) = \langle \mathbf{k}_s(s'_1), \mathbf{k}_s(s'_2)\rangle$. This curve is the normalized orthogonal projection of γ on $Vect^{\perp}(\mathbf{f}(s), \mathbf{T}(s))$ and consequently $s' \to \mathbf{k}_s(s')$ is not unit speed. In order to apply Corollary 2 to $Z_s = \sup_{s' \in J} W_s(s')$, it remains to determine $\|\mathbf{k}'_s(s')\|$ and the geodesic curvature $c_{g,s}(s')$ of γ_s.

Let us denote $\rho_{kl,s}(s') = D_{kl}r_s(s'_1, s'_2)|_{s'_1 = s'_2 = s'}$. We have $\|\mathbf{k}'_s(s')\| = \rho_{11,s}^{1/2}(s')$ and $c_{g,s}^2(s') = (\rho_{11,s}(s')\rho_{22,s}(s') - \rho_{12,s}^2(s'))/\rho_{11,s}^3(s') - 1$.

Theorem 3 (Lower bound) *Under conditions* (**C5**) – (**C9**) *and for* $n \geq 5$,

$f_Z(b) \geq m(b) =$

$$\frac{b}{2\pi}\int_0^L \tau_Y(s)(\tau_Y(s)+\tau_Y''(s))\exp\left(-\frac{b^2}{2}(\tau_Y^2(s)+\tau_Y'^2(s))\right)\left(1-\int_b^\infty M_s(b')db'\right)ds$$

$$-\frac{1}{2\pi}\int_0^L \tau_Y(s)c_g(s)\exp\left(-\frac{b^2}{2}(\tau_Y^2(s)+\tau_Y'^2(s))\right)\eta^{-1}(s)M_s(b)\,ds, \qquad (15)$$

for all $b > 0$, where

$$M_s(b) = \frac{b}{2\pi}\int_0^L \tau_s(s')\zeta_s(s')\exp\left(-\frac{b^2}{2}\left(\tau_s^2(s')+\frac{\tau_s'^2(s')}{\rho_{11,s}(s')}\right)\right)\Phi\left(\frac{b\zeta_s(s')}{c_{g,s}(s')}\right)\rho_{11,s}^{1/2}(s')ds'$$

$$+\frac{1}{2\pi}\int_0^L \tau_s(s')c_{g,s}(s')\exp\left(-\frac{b^2}{2}\left(\tau_s^2(s')+\frac{\tau_s'^2(s')}{\rho_{11,s}(s')}\right)\right)\varphi\left(\frac{b\zeta_s(s')}{c_{g,s}(s')}\right)\rho_{11,s}^{1/2}(s')ds'+\delta_{M_s}(b),$$

$$\delta_{M_s}(b) = \begin{cases} 0 & \gamma \textit{ closed,} \\ \tau_s(0)\varphi(b\tau_s(0))\Phi\left(\frac{b\tau_s'(0)}{\rho_{11,s}^{1/2}(0)}\right)+\tau_s(L)\varphi(b\tau_s(L))\left(1-\Phi\left(\frac{b\tau_s'(L)}{\rho_{11,s}^{1/2}(L)}\right)\right) & \textit{otherwise,} \end{cases}$$

and

$$\zeta_s(s') = \tau_s(s') - \tau_s'(s')\rho_{12,s}(s')/\rho_{11,s}^{3/2}(s') + \tau_s''(s')/\rho_{11,s}(s').$$

Remark. The corresponding expressions in terms of $t = \lambda^{-1}(s)$ are obtained by using the same transformations as in Corollary 2 and by replacing $\tau_s(s')$ by $\tau_t(t') = \beta_t(t')/\alpha_t(t')$ where $\alpha_t^2(t') = 1 - \langle \mathbf{g}(t), \mathbf{g}(t')\rangle^2 - \langle \mathbf{g}'(t), \mathbf{g}(t')\rangle^2/\|\mathbf{g}'(t)\|^2$, $\tau_s'(s')$ by $\tau_t'(t')/\|\mathbf{g}'(t')\|$, $\tau_s''(s')$ by $-\tau_t'(t')\langle \mathbf{g}'(t'), \mathbf{g}''(t')\rangle/\|\mathbf{g}'(t')\|^3 + \tau_t''(t')/\|\mathbf{g}'(t')\|^2$ and $r_s(s_1', s_2')$ by $r_t(t_1', t_2')$ with similar transformation rules for $\rho_{kl,s}(s')$.

Together with Corollary 2, Theorem 3 enables us to investigate $f_Z(b)$ and $P\{Z > a\}$ with a new accuracy. Both upper and lower bounds for the tail distribution of the maximum of large classes of Gaussian processes have been obtained by Samorodnitsky (1991) and Berman and Kôno (1989) among others. However, these results are asymptotic and involve unknown constants. The next theorem states the efficacy of our bounds. It shows that, in general, they are asymptotic to $f_Z(b)$ as $b \to \infty$. In addition, simple asymptotic formulae for $f_Z(b)$ are made explicit.

Theorem 4 *Under the conditions of Corollary 2 for $M(b)$ and of Theorem 3 for $m(b)$, and consequently for $f_Z(b)$,*

$$\begin{aligned} f_Z(b) &= e(b)(1+R(b)) \\ M(b) &\leq e(b)(1+R_M(b)), \quad R_M(b) > 0, \\ m(b) &\geq e(b)(1-R_m(b)), \quad R_m(b) > 0, \end{aligned}$$

as $b \to \infty$, where

(*i*) *if* γ *is closed, or* γ *is not closed and* $\min(\tau_Y(0), \tau_Y(L)) > \inf_{s\in J} \tau_Y(s)$,

$$e(b) = \frac{b}{2\pi}\int_0^L \tau_Y(s)(\tau_Y(s) + \tau_Y''(s))\exp\left(-\frac{b^2}{2}(\tau_Y^2(s) + \tau_Y'^2(s))\right) ds \qquad (16)$$

and $R(b) = R_M(b) = R_m(b) = O(\varphi(b\theta))$ *for some* $\theta > 0$*;*

(*ii*) *if* γ *is not closed and* $\tau_Y(s) \equiv 1$ *is constant,*

$$e(b) = \frac{b}{2\pi} L \exp(-b^2/2),$$

$R(b) = R_M(b) = O(b^{-1})$ *and* $R_M(b) = O(\varphi(b\theta))$ *for some* $\theta > 0$*;*

(*iii*) *if* γ *is not closed,* $\tau_Y(0) = \tau < \tau_Y(s)$ *for all* $s \neq 0$ *and* $q = \inf\{k \geq 1 : \tau_Y^{(k)}(0) = \tau^{(k)} > 0\}$ *is assumed finite, then*

1. $e(b) = b^{1-2/q}\exp(-b^2\tau^2/2)(2\pi)^{-1}\tau^{2-1/q}(\tau^{(k)})^{-1/q}q^{-1}\Gamma(q^{-1})\Gamma(q+1)^{1/q}$ *and* $R(b) = R_M(b) = R_m(b) = o(1)$ *if* $q \geq 3$*;*
2. $e(b) = \exp(-b^2\tau^2/2)(2\pi)^{-1/2}\tau(2^{-1}(\tau'')^{-1/2}(\tau+\tau'')^{1/2}+1)$, $R(b) = R_M(b) = o(1)$ *and* $R_m(b) = 1 - (2^{-1}(\tau'')^{-1/2}(\tau + \tau'')^{1/2} + 1)^{-1} - o(1)$ *if* $q = 2$*;*
3. $e(b) = \delta_M(b)$, $R(b) = o(1)$, $R_M(b) = O(\varphi(b\theta))$ *and* $R_m(b) = 1 - O(\varphi(b\theta))$ *for some* $\theta > 0$ *if* $q = 1$.

Remark. The remainders $R(b)$, $R_M(b)$ and $R_m(b)$ are of order $o(1)$ whenever we have used Laplace approximations (De Bruijn, 1962, p. 65) in (16) to determine the asymptotic formula for $f_Z(b)$ (see proof of Theorem 4). Such occurrences do not reflect an inaccuracy of our bounds.

Theorem 4 shows that, in general, $f_Z(b)$ can be approximated with (unexpected) high accuracy by the simple expressions $e(b)$. It also reveals that when the process $X(t)$ ($Y(s)$) reaches its maximum at the boundaries with high probability, the main contribution to the density $f_Z(b)$ is given by the additional term $\delta_Z(b)$. This phenomenon affects the good behavior of $m(b)$ since we have chosen to take 0 as lower bound for $\delta_Z(b)$ for sake of brevity. However, it is possible to improve $m(b)$ by introducing a term $0 < \delta_m(b) \leq \delta_Z(b)$ which corrects this imperfection (see Diebolt and Posse, 1994).

3 Extension to series representation

The results of Section 2 can be extended to the maximum Z of Gaussian processes of the form

$$X(t) = \tau_X^{-1}(t)U(t) \quad t \in I = [0, T],$$

where $\tau_X(t) > 0$, $U(t)$ is a centered Gaussian process with unit variance and finite covariance function

$$r_U(t_1,t_2) = \langle \mathbf{g}(t_1), \mathbf{g}(t_2) \rangle = \sum_{j=1}^{\infty} g_j(t_1) g_j(t_2).$$

The functions $\mathbf{g}(t) = (g_1(t), g_2(t), \ldots)$ parameterize a curve γ embedded in the unit sphere of ℓ^2. Let us assume that

(D1) the function $\tau_X(t)$ is in $C^3(I)$ and $r_U(t_1,t_2)$ has continuous partial derivatives $D_{kl} r_U(t_1,t_2)$ for $0 \le k,l \le 4$.

Let us denote $\rho_{kl,u}(t) = D_{kl} r_U(t_1,t_2)|_{t_1=t_2=t}$. We also assume that

(D2) $\rho_{11,u}(t) \neq 0$ for all $t \in I$;

(D3) $\tau_X(t) - \tau_X'(t)\rho_{12,u}(t)/\rho_{11,u}^{3/2}(t) + \tau_X''(t)/\rho_{11,u}(t) > 0$ for all $t \in I$;

(D4) the curve $\tilde{\gamma}$ parameterized by $\tilde{\mathbf{g}}(t) = \tau_X(t)\mathbf{g}(t) + \tau_X'(t)\mathbf{g}'(t)/\rho_{11,u}(t)$ has no self-intersection.

Under these conditions, we show the existence of a density for Z and give an upper bound. With the following assumptions we also derive a lower bound. Note that condition (D5) can be weakened since it is imposed by the sufficient conditions (D7) and (D8) (see Section 2).

(D5) The function $\tau_X(t)$ is in $C^4(I)$ and $r_U(t_1,t_2)$ has continuous partial derivatives $D_{kl} r_U(t_1,t_2)$ for $0 \le k,l \le 14$;

(D6) for all $t \in I$, $\beta_t(t') = \tau_X(t') - \tau_X(t) r_U(t,t') - \tau_X'(t) D_{10} r_U(t,t')/\rho_{11,u}(t) > 0$ for all $t' \neq t$, $t' \in I$;

(D7) for a.e. $t \in I$, the sequences $\mathbf{g}(t)$, $\mathbf{g}'(t)$, $\mathbf{g}(t')$, $\mathbf{g}'(t')$ and $\mathbf{g}''(t')$ are linearly independent for all $t' \neq t$, $t' \in I$;

(D8) for a.e. $t \in I$, the sequences $\mathbf{g}(t)$, $\mathbf{g}'(t)$, $\mathbf{g}(t')$, $\mathbf{g}'(t')$ and $\mathbf{g}(t'')$ are linearly independent for all $t' \neq t$, $t'' \neq t$, $t'' \neq t'$ and $t', t'' \in I$.

The length and geodesic curvature of γ are given by $L = \int_0^T \rho_{11,u}^{1/2}(t')dt'$ and $c_g^2(t) = (\rho_{11,u}(t)\rho_{22,u}(t) - \rho_{12,u}^2(t))/\rho_{11,u}^3(t) - 1$.

As in Section 2, γ has a unit speed parameterization $s = \lambda(t) = \int_0^t \rho_{11,u}^{1/2}(t')dt'$ under (D2). Therefore, $Z = \sup_{t\in I} X(t) = \sup_{s\in J} Y(s)$ where $Y(s) = \tau_Y^{-1}(s)V(s)$ with $\tau_Y(s) = \tau_X(\lambda^{-1}(s))$, $V(s) = U(\lambda^{-1}(s))$ a Gaussian process of variance one and covariance function $r_V(s_1,s_2) = \langle \mathbf{f}(s_1), \mathbf{f}(s_2)\rangle = \sum_{j=1}^{\infty} f_j(s_1)f_j(s_2)$, and $\mathbf{f}(s) = \mathbf{g}(\lambda^{-1}(s))$. We have $c_g^2(s) = \rho_{22,v}(s) - 1$ where $\rho_{kl,v}(s) = D_{kl} r_V(s_1,s_2)|_{s_1=s_2=s}$.

Finally, the functions $\alpha_s(s')$, $\tau_s(s')$, $\eta(s)$, $r_s(s'_1, s'_2)$, $\rho_{kl,s}(s')$, $c_{g,s}(s')$ and $\zeta_s(s')$ defined in Section 2 are also well-defined in the present context and we can show the following result.

Theorem 5 *(i) Under conditions* **(D1)** – **(D4)**, *Z has a density $f_Z(b)$ and $f_Z(b) \leq M(b)$ for all b, with $M(b)$ given by* (9)*;*

(ii) Under conditions **(D3)**, **(D5)** – **(D8)**, *$f_Z(b) \geq m(b)$ for all $b > 0$, with $m(b)$ given by* (15).

In addition, Theorem 4 still holds.

4 Proofs of results of Sections 2 and 3

4.1 Proof of Theorem 1

We give a detailed proof for the case γ closed and sketch the straightforward adaptation for the other case.

We need a workable description of the boundary ∂C_a of C_a, for $a \in \mathbb{R}$. Note first that C_a is the intersection of the closed half spaces $\{\mathbf{x} \in \mathbb{R}^n : Y(s,\mathbf{x}) \leq a\}$, $s \in J$. These half spaces have hyperplane boundaries given by

$$H(s;a) = \{\mathbf{x} \in \mathbb{R}^n : Y(s,\mathbf{x}) = a\}, \quad s \in J. \tag{17}$$

Lemma 6 shows that the surface ∂C_a is closely related to the hypersurface enveloped by the hyperplanes $H(s;a)$, $s \in J$.

Lemma 6 *If $C_a \neq \emptyset$, $\partial C_a = \{\mathbf{x} \in \mathbb{R}^n : \sup_{s \in J} Y(s,\mathbf{x}) = a\}$.*

Proof. $\partial C_a \supset \{\mathbf{x} \in \mathbb{R}^n : \sup_{s\in J} Y(s,\mathbf{x}) = a\}$. Let $\mathbf{x} \in \mathbb{R}^n$ satisfy $\sup_{s\in J} Y(s,\mathbf{x}) = a$. Then $\mathbf{x} \in C_a$. Let us prove that it does not intersect $int(C_a)$. Since the function $s \in J \to Y(s,\mathbf{x})$ is continuous on the compact interval J, it reaches its maximum. Therefore, there exists $s_0 \in J$ such that $\sup_{s\in J} Y(s,\mathbf{x}) = Y(s_0,\mathbf{x}) = a$. Let $\epsilon > 0$ be an arbitrary small positive real number. In the open ball $B(\mathbf{x}, 2\epsilon)$ with center $\mathbf{x}$ and radius 2ϵ, we can pick $\mathbf{y} = \mathbf{x} + \epsilon\mathbf{f}(s_0)$. Then $Y(s_0,\mathbf{y}) = a + \frac{\epsilon}{\tau_Y(s_0)} > a$. Hence $\mathbf{y} \notin C_a$ and $\mathbf{x} \in \partial C_a$.

$\partial C_a \subset \{\mathbf{x} \in \mathbb{R}^n : \sup_{s\in J} Y(s,\mathbf{x}) = a\}$. We will show that if $\sup_{s\in J} Y(s,\mathbf{x}) < a$, then $\mathbf{x} \in int(C_a)$. Since the function $Y(s,\mathbf{x})$ is continuous, there exists $s_0 \in J$ such that $\sup_{s\in J} Y(s,\mathbf{x}) = Y(s_0,\mathbf{x}) = b < a$. Then, for any sufficiently small $\epsilon > 0$ and for each $\mathbf{y} \in B(\mathbf{x},\epsilon)$, we have

$$\sup_{s\in J} Y(s,\mathbf{y}) \;\leq\; b + \sup_{s\in J} \frac{1}{\tau_Y(s)} \langle \mathbf{y} - \mathbf{x}, \mathbf{f}(s) \rangle$$

$$
\begin{aligned}
&\le \quad b + \sup_{s\in J} \frac{1}{\tau_Y(s)} \sup_{s\in J} \sup_{\mathbf{z}\in B(\mathbf{0},\epsilon)} |\langle \mathbf{z}, \mathbf{f}(s)\rangle| \\
&\le \quad b + \sup_{s\in J} \frac{1}{\tau_Y(s)} \,\epsilon < a.
\end{aligned}
$$

$\diamond$

Lemma 7 *$C_a = \Sigma_a \cap C_a$. where $\Sigma_a = \cup_{s\in J}\Pi_a(s)$, with $\Pi_a(s)$ denoting the affine subspace of dimension $n-2$ of $\mathbb{R}^n$ defined by the equations*

$$
\begin{cases} \langle \mathbf{x}, \mathbf{f}(s)\rangle &= a\tau_Y(s) \\ \langle \mathbf{x}, \mathbf{T}(s)\rangle &= a\tau_Y'(s). \end{cases} \tag{18}
$$

Proof. $\partial C_a \subset \Sigma_a \cap C_a$. From Lemma 6, it follows that for each $\mathbf{x} \in \partial C_a$, there exists $s_0 \in J$ such that $\sup_{s\in J} Y(s,\mathbf{x}) = Y(s_0,\mathbf{x}) = a$, i.e.

$$
\langle \mathbf{x}, \mathbf{f}(s_0)\rangle = a\tau_Y(s_0). \tag{19}
$$

At such a point s_0, the function $s \to Y(s,\mathbf{x})$ reaches its maximum, implying that

$$
\frac{\partial Y(s,\mathbf{x})}{\partial s}\Big|_{s=s_0} = -\frac{\tau_Y'(s_0)}{\tau_Y^2(s_0)}\langle \mathbf{x}, \mathbf{f}(s_0)\rangle + \frac{1}{\tau_Y(s_0)}\langle \mathbf{x}, \mathbf{T}(s_0)\rangle = 0. \tag{20}
$$

Plugging (19) into (20), we obtain (18).

$\partial C_a \supset \Sigma_a \cap C_a$. Let $\mathbf{x} \in \Sigma_a \cap C_a$. Then, $\sup_{s\in J} Y(s,\mathbf{x}) \le a$ (since $\mathbf{x} \in C_a$) and there exists $s_0 \in J$ such that $Y(s_0,\mathbf{x}) = a$ (since $\mathbf{x} \in \Sigma_a$). This implies that $\sup_{s\in J} Y(s,\mathbf{x}) = a$, which in turn implies that $\mathbf{x} \in \partial C_a$, by Lemma 6. $\diamond$

Lemma 8 (i) *The hypersurface Σ_a can be parameterized by*

$$
\mathbf{p}_a(s,u) = a(\tau_Y(s)\mathbf{f}(s) + \tau_Y'(s)\mathbf{T}(s)) + \sum_{j=1}^{n-2} u_j \mathbf{K}_j(s), \tag{21}
$$

with $s \in J$, $u = (u_1, \ldots, u_{n-2}) \in \mathbb{R}^{n-2}$.

(ii) *The hypersurface ∂C_a can be parameterized by $\mathbf{p}_a(s,u)$, with $s \in J$ and $u \in D_a(s)$, where $D_a(s)$ is the closed convex subset of $\mathbb{R}^{n-2}$ (possibly empty) defined by the set of inequalities* $\sup_{s'\in J} Y(s', \mathbf{p}_a(s,u)) \le a$.

Proof. (i) Since $\Sigma_a = \cup_{s\in J}\Pi_a(s)$, it suffices to prove that for each given $s_0 \in J$, $\mathbf{p}_a(s_0,u)$, $u \in \mathbb{R}^{n-2}$, parameterizes the affine subspace $\Pi_a(s_0)$. First, $\mathbf{p}_a(s_0,u) \in \Pi_a(s_0)$, since it satisfies (18). Second, the function $u \to \mathbf{p}_a(s_0,u)$ is an affine map from $\mathbb{R}^{n-2}$ to $\Pi_a(s_0)$. Since the dimension of the affine subspace $\Pi_a(s_0)$ of $\mathbb{R}^n$ is $n-2$, it remains to prove that $u \to \mathbf{p}_a(s_0,u)$ is an affine bijection. This is true since $\mathbf{K}_1(s_0), \ldots, \mathbf{K}_{n-2}(s_0)$ are linearly independent.

(ii) This follows directly from (i) and Lemma 7. $\diamond$

Lemma 9 *Let $s_0 \in J$ be given.*

(i) If $u \in D_a(s_0) \neq \emptyset$, then $d(a, s_0) - u_1 c_g(s_0) \geq 0$,

(ii) if $u \in int(D_c(s_0)) \neq \emptyset$ and $c_g(s_0) > 0$, then $d(a, s_0) - u_1 c_g(s_0) > 0$, where $d(a, s) = a(\tau_Y(s) + \tau_Y''(s))$.

Proof. (i) For each fixed $u \in \mathbb{R}^{n-2}$ and s_0, the function $h_{u,s_0}(s) = Y(s, \mathbf{p}_2(s_0, u))$, $s \in J$, is twice differentiable and $h'_{u,s_0}(s_0) = 0$. Furthermore, since $\mathbf{f}''(s) = c_g(s)\mathbf{K}(s) - \mathbf{f}(s)$, for all $s \in J$, $h''_{u,s_0}(s_0) = -(d(a, s_0) - u_1 c_g(s_0))/\tau_Y(s_0)$. If $u \in D_a(s_0) \neq \emptyset$, $h_{u,s_0}(s)$ reaches its maximum value at $s = s_0$, implying that $h''_{u,s_0}(s_0) \leq 0$.

(ii) Suppose that $u \in int(D_a(s_0)) \neq \emptyset$. Let us show by contradiction that $h''_{u,s_0}(s_0) < 0$. Otherwise, we would have $h''_{u,s_0}(s_0) = 0$ by (i). If $h''_{u,s_0}(s_0) = 0$, since $c_g(s_0) > 0$ and $u \in int(D_a(s_0))$, we can pick $v \in D_a(s_0)$ (close enough to u) such that $h''_{v,s_0}(s_0) > 0$ (by taking $v_1 > u_1$), which contradicts (i). $\diamond$

Remark. If $c_g(s_0) = 0$, the inequality $h''_{u,s_0}(s_0) \leq 0$ can hold iff $d(a, s_0) \geq 0$. This shows that in this case $D_a(s_0) = \emptyset$ for $d(a, s_0) < 0$. If $d(a, s_0) > 0$, then $h''_{u,s_0}(s_0) = -d(a, s_0)/\tau_Y(s_0) < 0$ for all $u \in \mathbb{R}^{n-2}$.

Let us define the C^1 function

$$\mathbf{p}(b, s, u) = \mathbf{p}_b(s, u), \quad b \in \mathbb{R}, \quad s \in J, \quad u = (u_1, \ldots, u_{n-2}) \in \mathbb{R}^{n-2}.$$

This function maps $V_a = \{(b, s, u) : b \leq a, s \in J \text{ and } u \in D_b(s)\} \subset \mathbb{R}^n$ onto $C_a \subset \mathbb{R}^n$ since $\cup_{b \leq a} \partial C_b = C_a$. Moreover, the restriction of $\mathbf{p}(b, s, u)$ to the open subset $V_a^0 = \{(b, s, u) : b < a, b \neq 0, s \in int(J) \text{ and } u \in int(D_b(s))\} \subset V_a$ maps V_a^0 onto a subset C_a^0 of C_a. In the following, we will assume that a is such that $V_a^0 \neq \emptyset$ (by convention, if $int(D_b(s)) = \emptyset$, then $(b, s, u) \notin V_a^0$).

Lemma 10 (i) $|\det D\mathbf{p}(b, s, u)| = \tau_Y(s)|d(b, s) - u_1 c_g(s)|$.

(ii) The function $\mathbf{p}(b, s, u)$ is a local C^1-diffeomorphism from V_a^0 onto C_a^0 and C_a^0 is an open subset of $\mathbb{R}^n$.

Proof. (i) It suffices to prove that Gram $D\mathbf{p} = \tau_Y^2(s)(d(b, s) - u_1 c_g(s))^2$. This is true since $\partial\mathbf{p}/\partial b = \tau_Y(s)\mathbf{f}(s) + \tau'_Y(s)\mathbf{T}(s)$, $\partial\mathbf{p}/\partial s = (d(b, s) - u_1 c_g(s))\mathbf{T}(s) + \sum_{j=1}^{n-2} e_j(s)\mathbf{K}_j(s)$, $\partial\mathbf{p}/\partial u_j = \mathbf{K}_j$, $j = 1, \ldots, n-2$ and the functions $e_j(s)$, $j = 1, \ldots, n-2$, need not be known since they disappear in the evaluation of Gram $D\mathbf{p}$.

(ii) It follows from (i) and Lemma 9 that for $(b, s, u) \in V_a^0$, $\det D\mathbf{p}(b, s, u) \neq 0$. Hence, the restriction of $\mathbf{p}$ to V_a^0 is a local C^1-diffeomorphism from V_a^0 onto its image C_a^0 and C_a^0 is an open subset of $\mathbb{R}^n$.

Lemma 11 $\mathbf{p}$ *is a one-to-one mapping from V_a^0 onto C_a^0.*

Proof. Since the hypersurfaces ∂C_b, $b \leq a$, partition C_a, it suffices to examine the case where

$$\mathbf{p}(b, s_0, u) = \mathbf{p}(b, s_1, v), \tag{22}$$

with (b, s_0, u) and (b, s_1, v) in V_a^0. Moreover, if $s_0 = s_1$, $u = v$ since $\mathbf{K}_j(s_0)$, $j = 1, \ldots, n-2$ are linearly independent.

It follows from (22) with $s_0 \neq s_1$, $s_0, s_1 \in int(J)$ that $Y(s_1, \mathbf{p}(b, s_0, u)) = b$. If there exists j $(1 \leq j \leq n-2)$ such that $\langle \mathbf{K}_j(s_0), \mathbf{f}(s_1) \rangle \neq 0$, a perturbation argument similar to the one used in the proof of Lemma 9 (ii) shows that $Y(s_1, \mathbf{p}(b, s_0, u)) < b$, since $u \in int(D_b(s_0))$. If $\langle \mathbf{K}_j(s_0), \mathbf{f}(s_1) \rangle = \langle \mathbf{K}_j(s_1), \mathbf{f}(s_0) \rangle = 0$ for all j $(1 \leq j \leq n-2)$, then $\mathbf{f}(s_1) \in Vect(\mathbf{f}(s_0), \mathbf{T}(s_0))$, $\mathbf{f}(s_0) \in Vect(\mathbf{f}(s_1), \mathbf{T}(s_1))$ and

$$b(\tau_Y(s_0)\mathbf{f}(s_0) + \tau_Y'(s_0)\mathbf{T}(s_0)) = b(\tau_Y(s_1)\mathbf{f}(s_1) + \tau_Y'(s_1)\mathbf{T}(s_1))$$

which is impossible since $b \neq 0$ and condition (**C4**) holds. $\diamond$

Lemma 12

$$\mu_n(C_a^0) = \int_{V_a^0} \tau_Y(s)(d(b,s) - u_1 c_g(s))\varphi_n(\mathbf{p}(b, s, u))\, du_1 \ldots du_{n-2}\, ds\, db. \qquad (23)$$

Proof. According to Lemmas 10 and 11, the function $\mathbf{p}$ is a C^1-diffeomorphism from V_a^0 onto C_a^0. Moreover, according to Lemmas 9 and 10, $\det D\mathbf{p}(b, s, u) > 0$ for all $(b, s, u) \in V_a^0$. Then, (23) results from the change-of-variable $\mathbf{x} = \mathbf{p}(b, s, u)$ applied to the integral $\mu_n(C_a^0) = \int_{C_a^0} \varphi_n(\mathbf{x}) d\mathbf{x}$. $\diamond$

Since $D_b(s)$ is a convex of $\mathbb{R}^{n-2}$, the Lebesgue measure in $\mathbb{R}^{n-2}$ of $\partial D_b(s)$ is zero. Therefore, the Lebesgue measure of $V_a \backslash V_a^0$ is zero and we can replace V_a^0 by V_a in (23). Moreover, $C_a \backslash C_a^0$ has Lebesgue measure zero since $C_a \backslash C_a^0 = \mathbf{p}(V_a) \backslash \mathbf{p}(V_a^0) \subset \mathbf{p}(V_a \backslash V_a^0)$ and $\mathbf{p}(b, s, u)$ is a C^1-function from $\mathbb{R}^n$ to $\mathbb{R}^n$. Consequently, $\mu_n(C_a) = \mu_n(C_a^0)$ which, with (5) and (6), concludes the proof of Theorem 1. $\diamond$

Suppose now that γ is not closed. Lemma 6 still holds. In Lemma 7, we have to replace Σ_a by $\Sigma_a' = (\cup_{s \in int(J)} \Pi_a(s)) \cup H(0, a) \cup H(L, a)$ where $H(l, a)$ is defined in (17). Then ∂C_a can be partitioned as $\partial C_a = \partial C_{a,int} \cup \partial C_{a,0} \cup \partial C_{a,L}$ where $\partial C_{a,int} = (\cup_{s \in int(J)} \Pi_a(s)) \cap C_a$, $\partial C_{a,l} = H(l, a) \cap C_a$, $l = 0, L$. Lemmas 8–12 can be applied without modification to $C_{a,int} = \cup_{b \leq a} \partial C_{a,int}$. The additional term $\delta_Z(b)$ is obtained from the boundaries $C_{a,l} = \cup_{b \leq a} \partial C_{a,l}$, $l = 0, L$. Indeed, for $l = 0, L$, $\partial C_{a,l}$ can be parameterized by $\mathbf{p}_{b,l}(v) = b\tau_Y(l)\mathbf{f}(l) + v_{n-1}\mathbf{T}(l) + \sum_{j=1}^{n-2} v_j \mathbf{K}_j(l)$ with $v \in G_b(l) = \{v \in \mathbb{R}^{n-1} : \sup_{s' \in J} \tau_Y(s')^{-1} \langle \mathbf{p}_{b,l}(v), \mathbf{f}(s') \rangle \leq b\}$. Since Gram $D\mathbf{p}_{b,l} = \tau_Y^2(l)$, we have $\psi_b(\partial C_{b,l}) = \int_{v \in G_a(l)} \varphi_n(\mathbf{p}_{b,l}(v)) \tau_Y(l) dv$. $\diamond$

4.2 Proof of Theorem 3

Let J^* be a subset of J such that $J \backslash J^*$ has Lebesgue measure zero and for each $s \in J^*$, $\mathbf{f}(s)$, $\mathbf{T}(s)$, $\mathbf{f}(s')$, $\mathbf{T}(s')$ and $\mathbf{N}(s')$ are linearly independent for all $s' \neq s$, $s' \in J$, and $\mathbf{f}(s)$, $\mathbf{T}(s)$, $\mathbf{f}(s')$, $\mathbf{T}(s')$ and $\mathbf{f}(s'')$ are linearly independent for all $s' \neq s$, $s'' \neq s$, $s'' \neq s'$ and $s', s'' \in J$.

For each $s \in J^*$, $\mu_{n-2}(D_b(s))$ in formula (10) can be interpreted as $P(Z_s \leq b)$ where $Z_s = \sup_{s' \in J} W_s(s')$,

$$W_s(s') = \begin{cases} \sum_{j=1}^{n-2} \omega_j \langle \mathbf{K}_j(s), \mathbf{f}(s') \rangle / \beta_s(s') & s' \neq s \\ \omega_1 c_g(s) / (\tau_Y(s) + \tau_Y''(s)) & s' = s, \end{cases}$$

with $\boldsymbol{\Omega} = (\omega_1, \ldots, \omega_{n-2})$ a Gaussian r.v. with zero mean and identity covariance matrix. With this definition, $W_s(s')$ is defined and continuous on J. $Var(W_s(s')) = \alpha_s^2(s')/\beta_s^2(s')$ for $s' \neq s$ and $Var(W_s(s)) = c_g^2(s)/(\tau_Y(s) + \tau_Y''(s))^2$. Moreover, $W_s(s')$ is of the form (4) and satisfies conditions (C1)–(C4): $W_s(s') = \tau_s^{-1}(s')\langle \boldsymbol{\Omega}, \mathbf{k}_s(s') \rangle$, where $\tau_s(s') = (Var(W_s(s')))^{-1/2}$ and $\mathbf{k}_s(s') = (k_{s,1}(s'), \ldots, k_{s,n-2}(s'))$ with $k_{s,j}(s') = \langle \mathbf{K}_j(s), \mathbf{f}(s') \rangle / \alpha_s(s')$ for $s' \neq s$, $k_{s,1}(s) = 1$ and $k_{s,j}(s) = 0$, $j \geq 2$. Therefore, we can apply Corollary 2 to Z_s to obtain an upper bound for $1 - \mu_{n-2}(D_b(s))$.

The above interpretation can also be used to derive an upper bound for the second term in formula (10). Indeed, by Stokes' Theorem (Berger and Gostiaux, 1988, page 195)

$$\left| \int_{D_b(s)} u_1 \varphi_{n-2}(u)\, du_1 \ldots du_{n-2} \right| = \left| \oint_{\partial D_b(s)} \varphi_{n-2}(u)\, du_2 \wedge \ldots \wedge du_{n-2} \right| \leq \int_{\partial D_b(s)} \varphi_{n-2}\, dV,$$

where dV denotes the canonical volume element of the manifold $\partial D_b(s)$ (Berger and Gostiaux, 1988, page 203). A straightforward adaptation of Lemmas 6–9 yields

$$\int_{\partial D_b(s)} \varphi_{n-2}\, dV = \int_{(s',v) \in \mathbf{p}_{b,s}^{-1}(\partial D_b(s))} \varphi_{n-2}(\mathbf{p}_{b,s}(s',v))\, \mathrm{Gram}^{1/2} D\mathbf{p}_{b,s}(s',v)\, ds'\, dv,$$

where $s' \in J$, $v \in \mathbb{R}^{n-4}$ and $(s',v) \to \mathbf{p}_{b,s}(s',v)$ defines a parameterization of $\partial D_b(s)$ analogous to (21). From (6) and (7) applied to the process $W_s(s')$, it follows that

$$\int_{\partial D_b(s)} \varphi_{n-2}\, dV \leq \frac{1}{\inf\limits_{s' \in J} \tau_s(s')} f_{Z_s}(b) = \eta^{-1}(s)\, f_{Z_s}(b),$$

where $f_{Z_s}(b)$ is the density of Z_s. ◊

4.3 Proof of Theorem 4

(i) Assume γ closed and let denote

$$e_1(b) = \frac{b}{2\pi} \int_0^L \tau_Y(s)(\tau_Y(s) + \tau_Y''(s)) \exp\left(-\frac{b^2}{2}(\tau_Y^2(s) + \tau_Y'^2(s)) \right) ds.$$

By (9), $M(b) = e_1(b) +$

$$\frac{b}{2\pi} \int_0^L \tau_Y(s)(\tau_Y(s) + \tau_Y''(s)) \exp\left(-\frac{b^2}{2}(\tau_Y^2(s) + \tau_Y'^2(s)) \right) H\left(\frac{b(\tau_Y(s) + \tau_Y''(s))}{c_g(s)} \right) ds,$$

where $0 < H(x) = x^{-1}\varphi(x) - (1-\Phi)(x) \le x^{-3}\varphi(x)$ for all $x > 0$ and $H(\infty) = 0$. Since $c_g(s)$ is continuous and $\tau_Y(s) + \tau_Y''(s)$ is continuous and positive on J, it follows that $\theta_1 = \inf_{s\in J}(\tau_Y(s) + \tau_Y''(s))/c_g(s) > 0$ and

$$M(b) - e_1(b) \le e_1(b)(b\theta_1)^{-3}\varphi(b\theta_1).$$

By (15), $m(b) = e_1(b) - A - B$, where A involves $\int_b^\infty M_s(b')db'$ and B involves $M_s(b)$. Therefore, it suffices to examine the asymptotic behavior of $M_s(b)$. Since $\eta(s) > 0$,

$$M_s(b) \le \frac{1}{2\pi}\exp\left(-\frac{b^2\eta^2(s)}{2}\right)\int_0^L \tau_s(s')(b\zeta_s(s') + c_{g,s}(s')(2\pi)^{-1/2})\rho_{11,s}^{1/2}(s')\,ds'.$$

Using Taylor expansions of sufficient order, it can be shown that the functions $(s,s') \in J \times J \to \tau_s(s'), \tau_s'(s'), \tau_s''(s')$, $\rho_{kl,s}(s')$, $0 \le k,l \le 2$, and $c_{g,s}(s')$ are continuous. Therefore, their supremum over $J \times J$ is finite. Moreover, by the continuity and positivity of $\tau_s(s')$ as a function of $(s,s') \in J \times J$, $\theta_2 = \inf_{s\in J}\eta(s) = \inf_{(s,s')\in J\times J}\tau_s(s') > 0$. It follows that $M_s(b) \le C_1\, b\,\varphi(b\theta_2)$ for all $s \in J$ and all $b > 0$, hence $\int_b^\infty M_s(b')\,db' \le C_2\,\varphi(b\theta_2)$. Consequently, $A \le e_1(b)\,C_3\,\varphi(b\theta_2)$ and $B \le e_1(b)\,C_4\,\varphi(b\theta_2)$. From the asymptotic behavior of $M(b)$ and $m(b)$, it results that $e_1(b) = e(b)$.

If γ is not closed and $\min(\tau_Y(0), \tau_Y(L)) > \inf_{s\in J}\tau_Y(s)$,

$$\frac{\varphi(b\tau_Y(0))}{e_1(b)} = C_5\left(\int_0^L \tau_Y(s)(\tau_Y(s) + \tau_Y''(s))\exp\left(\frac{b^2}{2}(\tau_Y^2(0) - \tau_Y^2(s) - {\tau_Y'}^2(s))\right)ds\right)^{-1}.$$

For $\epsilon > 0$ small enough, the subset $J_\epsilon = \{s \in J : \tau_Y^2(0) - \tau_Y^2(s) - {\tau_Y'}^2(s) \ge \epsilon^2\}$ of J contains a nonempty interval (around a global minimum of $\tau_Y^2(s) + {\tau_Y'}^2(s)$). Therefore, it has positive Lebesgue measure. For such an $\epsilon > 0$,

$$\int_0^L \tau_Y(s)(\tau_Y(s) + \tau_Y''(s))\exp\left(\frac{b^2}{2}(\tau_Y^2(0) - \tau_Y^2(s) - {\tau_Y'}^2(s))\right)ds$$
$$\ge \exp(b^2\epsilon^2/2)\int_{J_\epsilon}\tau_Y(s)(\tau_Y(s) + \tau_Y''(s))\,ds = C_6\,\varphi^{-1}(b\epsilon).$$

Hence, $\varphi(b\tau_Y(0)) = e_1(b)\,O(\varphi(b\theta_3))$, i.e. $\delta_M(b) = e_1(b)\,O(\varphi(b\theta_3))$. To complete the proof, it remains to show that $\sup_{s\in J}\delta_{M_s}(b) = e_1(b)\,O(\varphi(b\theta_4))$. This follows from the continuity and positivity of $s \to \tau_s(0)$ and $\tau_s(L)$.

(ii) Straightforward.

(iii) First, note that $\tau^{(q)} = \tau_Y^{(q)}(0) > 0$. Since $m(b) - e_1(b) = e_1(b)O(\varphi(b\theta_5))$ by (i), it suffices to study the asymptotic behavior of $e_1(b)$ and $\delta_M(b)$. The result follows directly from the Laplace approximation (De Bruijn, 1962, p. 65) of $e_1(b)$ using a Taylor expansion of order q of $\tau_Y^2(s) + {\tau_Y'}^2(s)$ around $s = 0$. $\diamond$

4.4 Proof of Theorem 5

Lemma 13 shows that the process $X(t)$ admits the representation (1). Therefore, by truncation and renormalization, we can construct a sequence of Gaussian processes $X_n(t)$ of the form (4) which converge to $X(t)$. Moreover, under (D1)–(D4), $X_n(t)$ satisfies (C1)–(C4) for all n sufficiently large. Similarly, under (D3), (D5)–(D8), $X_n(t)$ satisfies (C5)–(C9) for n large enough. Hence, the density $f_{Z_n}(b)$ of $Z_n = \sup_{t\in I} X_n(t)$ has an upper bound $M_n(b)$ of the form (9) and a lower bound $m_n(b)$ of the form (15).

We will show that $M_n(b) \to M(b)$ for all b, $m_n(b) \to m(b)$ for all $b > 0$, and the sequence $\{f_{Z_n}\}$ is weakly relatively compact in $L^1(\mathbb{R})$. With an equality due to Dmitrovskii (Lifshits, 1986), this implies that Z has a density $f_Z(b)$ which is the limit of $f_{Z_n}(b)$ and satisfies $m(b) \le f_Z(b) \le M(b)$.

Lemma 13 *Under condition* (**D1**),

(*i*) *there exists a representation* $r_X^{-1}(t)\sum_{j=1}^{\infty}\xi_j g_j(t)$ *of* $X(t)$, *where the r.v.'s* ξ_j, $j \ge 1$, *are independent, identically distributed* $\mathcal{N}(0,1)$;

(*ii*) *the functions* $g_j(t)$, $j \ge 1$, *are in* $C^3(I)$;

(*iii*) $\sup_{(t_1,t_2)\in I\times I} |D_{kl} r_U(t_1,t_2) - \sum_{j=1}^{n} g_j^{(k)}(t_1) g_j^{(l)}(t_2)| \to 0$ *as* $n \to \infty$, $0 \le k, l \le 3$.

Proof. (*i*) The Gaussian process $\tilde{U}_n(t) = \sum_{j=1}^{n}\xi_j g_j(t)$, $t \in I$, is centered and has covariance function

$$r_{\tilde{U}_n}(t_1,t_2) = \sum_{j=1}^{n} g_j(t_1) g_j(t_2), \quad t_1, t_2 \in I.$$

For each $t \in I$, the sequence $\{\tilde{U}_n(t) : n \ge 1\}$ is Cauchy in $L^2(\Omega, \mathcal{A}, P)$, since, for all p, q such that $1 \le p < q$, we have

$$E\Big(\sum_{j=p+1}^{q} \xi_j g_j(t)\Big)^2 = \sum_{j=p+1}^{q} g_j^2(t) \to 0,$$

as $1 \le p < q \to \infty$. Therefore, this sequence converges in quadratic mean to some Gaussian r.v. $\tilde{U}_t$ with mean zero and variance

$$\lim_{n\to\infty} \sum_{j=1}^{n} g_j^2(t) = \sum_{j=1}^{\infty} g_j^2(t) = r_U(t,t).$$

Indeed, $|E^{1/2}\tilde{U}_t^2 - E^{1/2}\tilde{U}_n^2(t)| \le E^{1/2}(\tilde{U}_t - \tilde{U}_n(t))^2 \to 0$ as $n \to \infty$. Similarly, for any $d \ge 1$ and any $(t_1, \ldots, t_d) \in I^d$, the sequence $\{U_n(t_1), \ldots, U_n(t_d) : n \ge 1\}$ converges in quadratic mean to a Gaussian $\mathbb{R}^d$-valued r.v. $\tilde{U}_{t_1,\ldots,t_d}$ with mean $(0, \ldots, 0)$ and covariance matrix $[r_U(t_l, t_k)]_{1\le k,l\le d}$.

The family of the distributions of the r.v.'s $\tilde{U}_{t_1,\ldots,t_d}$, $(t_1,\ldots,t_d) \in I^d$, $d \geq 1$, satisfies the symmetry and consistency conditions (Adler, 1981, pp. 13–14). By Kolmogorov's Theorem, there exists a Gaussian process $U(t)$ such that $U(t) = \tilde{U}_t$ a.s. for all $t \in I$. This process can be written as $\sum_{j=1}^{\infty} \xi_j g_j(t)$. This means that for each $t \in I$, $U(t)$ is the limit of $\sum_{j=1}^{n} \xi_j g_j(t)$ in quadratic mean (the same argument applies to $(U(t_1),\ldots,U(t_d))$). Moreover, by Cauchy-Schwarz Inequality,

$$\begin{aligned} &|E(U(t_1)U(t_2)) - E(\tilde{U}_n(t_1)\tilde{U}_n(t_2))| \\ &\quad \leq |E(U(t_1)U(t_2)) - E(\tilde{U}_n(t_1)U(t_1))| + |E(\tilde{U}_n(t_1)U(t_1)) - E(\tilde{U}_n(t_1)\tilde{U}_n(t_2))| \\ &\quad \leq E^{1/2}U^2(t_2)E^{1/2}(U(t_1) - \tilde{U}_n(t_1))^2 + E^{1/2}\tilde{U}_n^2(t_1)E^{1/2}(U(t_2) - \tilde{U}_n(t_2))^2 \to 0 \end{aligned}$$

as $n \to \infty$, since $U(t)$ is the limit in quadratic mean of $\tilde{U}_n(t)$, $EU^2(t_2) < \infty$ and $E\tilde{U}_n^2(t_1) \to EU^2(t_1) < \infty$ as $n \to \infty$. This implies that $E(U(t_1)U(t_2)) = \lim_{n\to\infty} r_{\tilde{U}_n}(t_1,t_2) = r_U(t_1,t_2)$.

(ii) From (i), we deduce that $g_j(t) = E(U(t)\,\xi_j)$ for all $j \geq 1$ and all $t \in I$. Besides, by Theorem 2.2.2 in Adler (1981), the process $U(t)$, $t \in I$, is 4-times differentiable in quadratic mean, under condition (D1). Therefore,

$$\frac{g_j(t+h) - g_j(t)}{h} = E(\dot{U}(t)\,\xi_j) + E\left(\left(\frac{U(t+h) - U(t)}{h} - \dot{U}(t)\right)\xi_j\right)$$

where $\dot{U}(t)$ denotes the quadratic mean derivative of $U(t)$. By Cauchy-Schwarz Inequality and definition of $\dot{U}(t)$, we have $g_j'(t) = E(\dot{U}(t)\,\xi_j)$. Similarly, we can show that for all $0 \leq k \leq 4$, $g_j(t)$ has a k-th order derivative

$$g_j^{(k)}(t) = E(\dot{U}^{(k)}(t)\,\xi_j) \tag{24}$$

where $\dot{U}^{(k)}(t)$ denotes the k-th quadratic mean derivative of $U(t)$. Finally, $g_j^{(k)}(t)$ is continuous for all $0 \leq k \leq 3$.

(iii) For $k = 0$, since $g_j(t)$ is continuous for all $j \geq 1$ and the sequence $\{\sum_{j=1}^{n} g_j^2(t) : n \geq 1\}$ is increasing, it follows from Dini's Theorem that

$$\sup_{t\in I} \sum_{j=n+1}^{\infty} g_j^2(t) = \sup_{t\in I}\left| r_U(t,t) - \sum_{j=1}^{n} g_j^2(t)\right| \to 0 \quad (n \to \infty).$$

Since $(\sum_{j=n+1}^{q} g_j(t_1)g_j(t_2))^2 \leq \sum_{j=n+1}^{q} g_j^2(t_1)\sum_{j=n+1}^{q} g_j^2(t_2)$ for all $q > n$, letting $q \to \infty$, we obtain

$$\sup_{(t_1,t_2)\in I\times I}\left|\sum_{j=n+1}^{\infty} g_j(t_1)g_j(t_2)\right| = \sup_{(t_1,t_2)\in I\times I}\left| r_U(t_1,t_2) - \sum_{j=1}^{n} g_j(t_1)g_j(t_2)\right| \to 0\,(n \to \infty).$$

For $D_{kl}r_U(t_1,t_2)$, $0 \leq k,l \leq 3$, by Parseval's Inequality applied to (24),

$$\sum_{j=1}^{\infty} g_j^{(k)2}(t) \leq E\dot{U}^{(k)2}(t) < \infty,$$

for all $t \in I$ and all $0 \le k, l \le 4$. Therefore, given $0 \le k, l \le 3$, we can construct a Gaussian process $\sum_{j=1}^{\infty} \xi_j g_j^{(k)}(t)$, $t \in I$, as in (i). Moreover, since the functions $g_j^{(k)}(t)$, $j \ge 1$, $0 \le k \le 3$, are continuous by (ii), by Dini's Theorem,

$$\sup_{t \in I} \sum_{j=n+1}^{\infty} g_j^{(k)2}(t) \to 0 \quad (n \to \infty), \tag{25}$$

for $0 \le k \le 3$, and the covariance function of this process is the uniform limit of $\sum_{j=1}^{n} g_j^{(k)}(t_1) g_j^{(k)}(t_2)$ as $n \to \infty$ for $(t_1, t_2) \in I \times I$. Finally, it results from (25) that for each given $t_2 \in I$, the series $\sum_{j=1}^{\infty} g_j(t_1) g_j(t_2)$ of functions of t_1 can be derived term-by-term 3-times. Hence,

$$D_{k0} r_U(t_1, t_2) = \sum_{j=1}^{\infty} g_j^{(k)}(t_1) g_j(t_2), \quad 0 \le k \le 3. \tag{26}$$

Keeping t_1 constant in (26), it results from (25) that the series $\sum_{j=1}^{\infty} g_j^{(k)}(t_1) g_j(t_2)$ of functions of t_2 can be derived term-by-term 3-times. Hence,

$$D_{kl} r_U(t_1, t_2) = \sum_{j=1}^{\infty} g_j^{(k)}(t_1) g_j^{(l)}(t_2), \quad 0 \le k, l \le 3.$$

Moreover, by Cauchy-Schwarz Inequality and (25) the series $\sum_{j=1}^{n} g_j^{(k)}(t_1) g_j^{(l)}(t_2)$ converges uniformly. ◇

Remark. Under (D1), the existence of functions $g_j(t)$ such that $r_U(t_1, t_2) = \sum_{j=1}^{\infty} g_j(t_1) g_j(t_2)$ results from Mercer's Theorem (see, e.g. Adler, 1981, Theorem 3.3.1), where $g_j(t) = \lambda_j^{1/2} \phi_j(t)$, $j \ge 1$, where the λ_j's are the eigenvalues and the $\phi_j(t)$'s are the eigenfunctions of the integral operator with kernel $r_U(t_1, t_2)$.

Let $\mathbb{P}_n$: $\ell^2 \to \ell^2$ denote the orthogonal projection of ℓ^2 onto $\mathbb{R}^n \times \{0, \ldots\}$, defined by $\mathbb{P}_n \mathbf{x} = (x_1, \ldots, x_n, 0, \ldots)$ for $\mathbf{x} \in \ell^2$.

Lemma 14 (i) *Under condition* **(D1)**, $\|\mathbb{P}_n \mathbf{g}(t)\| = 1 + \epsilon_n(t)$ *with* $\epsilon_n^{(k)}(t) \to 0$ *as* $n \to \infty$ *uniformly for* $t \in I$, *for* $0 \le k \le 3$;

(ii) *Under conditions* **(D1)** $-$ **(D2)**, $\|\mathbb{P}_n \mathbf{g}'(t)\| \to \|\mathbf{g}'(t)\|$ *uniformly for* $t \in I$ *as* $n \to \infty$ *and there exists* N_1 *such that* $\inf_{t \in I} \|\mathbb{P}_n \mathbf{g}'(t)\| > 0$ *for all* $n \ge N_1$.

Proof. (i) It suffices to prove a similar result for $\|\mathbb{P}_n \mathbf{g}(t)\|^2 = \sum_{j=1}^{n} g_j^2(t) = 1 - \sum_{j=n+1}^{\infty} g_j^2(t)$. For each $0 \le k \le 3$, the k-th derivative of $\sum_{j=n+1}^{\infty} g_j^2(t)$ is a linear combination of functions of the form $\sum_{j=n+1}^{\infty} g_j^{(l_1)}(t) g_j^{(l_2)}(t)$ with $0 \le l_1, l_2 \le k$. By Lemma 13, such functions converge to zero uniformly for $t \in I$.

(ii) The first statement is a straightforward consequence of Lemma 13. Since $\|\mathbf{g}'(t)\|$ is a continuous and positive function of $t \in I$ and I is compact, $\delta =$

$\inf_{t\in I}\|\mathbf{g}'(t)\| > 0$. We can choose N_1 such that $\sup_{t\in I}|\,\|\mathbf{g}'(t)\| - \|\mathbb{P}_n\mathbf{g}'(t)\|\,| < \delta/2$ for all $n \geq N_1$. Then $\inf_{t\in I}\|\mathbb{P}_n\mathbf{g}'(t)\| \geq \delta/2 > 0$ for all $n \geq N_1$. $\diamond$

It follows from Lemma 14 that

$$\mathbf{g}_n(t) = \frac{\mathbb{P}_n\mathbf{g}(t)}{\|\mathbb{P}_n\mathbf{g}(t)\|} = (g_{n,1}(t), \ldots, g_{n,n}(t), 0, \ldots) \quad t \in I,$$

is well-defined for all $n \geq N_1$ and $\mathbf{g}_n^{(k)}(t) \to \mathbf{g}^{(k)}(t)$ uniformly for $0 \leq k \leq 3$. The functions $\mathbf{g}_n(t)$, $t \in I$, parameterize a curve γ_n on the unit sphere of $\mathbb{R}^n \times \{0, \ldots\} \subset \ell^2$. Moreover, there exists N_2 such that $\inf_{t\in I}\|\mathbf{g}_n'(t)\| > 0$ for all $n \geq N_2$. The corresponding unit speed parameterization of γ_n is defined by $\mathbf{f}_n(s_n) = (f_{n,1}(s_n), \ldots, f_{n,n}(s_n), 0, \ldots) = \mathbf{g}_n(\lambda_n^{-1}(s_n))$, where $s_n = \lambda_n(t) = \int_0^t \|\mathbf{g}_n'(t')\|dt' \in J_n = [0, L_n]$. With this notation $Z_n = \sup_{t\in I} X_n(t) = \sup_{s_n \in J_n} Y_n(s_n)$ where $X_n(t) = \tau_X^{-1}(t)U_n(t)$, $U_n(t) = \sum_{j=1}^n \xi_j g_{n,j}(t)$, $Y_n(s_n) = \tau_{Y_n}^{-1}(s_n)V_n(s_n)$, $V_n(s_n) = \sum_{j=1}^n \xi_j f_{n,j}(s_n) = U_n(\lambda_n^{-1}(s_n))$ and $\tau_{Y_n}(s_n) = \tau_X(\lambda_n^{-1}(s_n))$.

Lemma 15 *Under conditions* $(\mathbf{D1}) - (\mathbf{D2})$,

(*i*) $\lambda_n^{(k)}(t) \to \lambda^{(k)}(t)$ *uniformly for* $t \in I$, *for* $0 \leq k \leq 3$;

(*ii*) $(\lambda_n^{-1})^{(k)}(s) \to (\lambda^{-1})^{(k)}(s)$ *uniformly for* $s \in J_n$, *for* $0 \leq k \leq 3$;

(*iii*) $\mathbf{f}_n^{(k)}(s) \to \mathbf{f}^{(k)}(s)$ *uniformly for* $s \in J_n$, *for* $0 \leq k \leq 3$;

(*iv*) $\tau_{Y_n}^{(l)}(s) \to \tau_Y^{(l)}(s)$ *uniformly for* $s \in J_n$, *for* $0 \leq l \leq 3$;

(*v*) $c_{g,n}(s) \to c_g(s)$ *uniformly for* $s \in J_n$.

Proof. (*i*) By Lemma 14, $\mathbf{g}_n'(t) \to \mathbf{g}'(t)$ uniformly. Therefore, $\lambda_n'(t) = \|\mathbf{g}'(t)\| \to \lambda'(t) = \|\mathbf{g}'(t)\|$ uniformly and $\lambda_n(t) = \int_0^t \|\mathbf{g}'(t')\|dt' \to \lambda(t) = \int_0^t \|\mathbf{g}'(t'\|dt'$ uniformly. As a consequence, $L_n = \lambda_n(T) \to L = \lambda(T)$. Moreover, the k-th derivative of $\lambda_n'(t)$ (respectively $\lambda'(t)$) is the product of a polynomial in $\langle \mathbf{g}_n^{(l_1)}(t), \mathbf{g}_n^{(l_2)}(t)\rangle$ (respectively $\langle \mathbf{g}^{(l_1)}(t), \mathbf{g}^{(l_2)}(t)\rangle$), $0 \leq l_1, l_2 \leq k$, and $\|\mathbf{g}_n'(t)\|^{-(2k-1)}$ (respectively $\|\mathbf{g}'(t)\|^{-(2k-1)}$). Therefore, the result follows from Lemma 14.

(*ii*) For $k = 0$, this is a direct consequence of (*i*). For $k \geq 1$, the result follows also from (*i*) since the k-th derivative of $\lambda_n^{-1}(s)$ (resp. $\lambda^{-1}(s)$) is an expression involving the successive derivatives of $\lambda_n(t)$ (resp. $\lambda(t)$) at $\lambda_n^{-1}(s)$ (resp. $\lambda^{-1}(s)$).

(*iii*) This is a direct consequence of Lemma 13 and (*ii*).

(*iv*) This follows directly from (*i*), (*ii*) and the uniform continuity of $\tau_X(t)$ and its derivatives over the compact I.

(*v*) This follows from (*iii*). $\diamond$

If (D1) is replaced by (D5) in Lemmas 14 and 15, all the results hold for $0 \leq k \leq 13$ and $0 \leq l \leq 4$.

Lemma 16 *Under conditions* $(\mathbf{D1}) - (\mathbf{D4})$,

(i) *There exists* N_3 *such that for all* $n \geq N_3$, $\delta = \inf_{s \in J_n}(\tau_{Y_n}(s) + \tau''_{Y_n}(s)) > 0$;

(ii) *there exists* N_4 *such that for all* $n \geq N_4$, *the curve* $\tilde{\gamma}_n$ *parameterized by* $\tilde{\mathbf{f}}_n(s) = \tau_{Y_n}(s)\mathbf{f}_n(s) + \tau'_{Y_n}(s)\mathbf{f}'_n(s)$, $s \in J_n$, *has no self-intersection.*

Proof. (i) The proof is similar to the proof of Lemma 14 (ii).

(ii) We first show that there exists N'_4 and $\epsilon > 0$ such that $\tilde{\mathbf{f}}_n(s') \neq \tilde{\mathbf{f}}_n(s)$ for all $s, s' \in J_n$, $|s' - s| < \epsilon$ and $n \geq N'_4$. By a quadratic Taylor expansion of $\tilde{\mathbf{f}}_n(s')$ around s, $\tilde{\mathbf{f}}_n(s + h) = \tilde{\mathbf{f}}_n(s) + h\tilde{\mathbf{f}}'_n(s) + \mathbf{R}_n(s, h)$, where $\|\mathbf{R}_n(s, h)\| \leq Ch^2$ and $C = \sup_{n \geq N_2} \sup_{s \in J} \|\tilde{\mathbf{f}}''_n(s)\| < \infty$ in view of Lemmas 13–15. Since $\tilde{\mathbf{f}}'_n(s) = (\tau_{Y_n}(s) + \tau''_{Y_n}(s))\mathbf{f}'_n(s) + \tau'_{Y_n}(s)(\mathbf{f}_n(s) + \mathbf{f}''_n(s))$, $\|\tilde{\mathbf{f}}'_n(s)\| \geq \tau_{Y_n}(s) + \tau''_{Y_n}(s) \geq \delta > 0$ for all $s \in J_n$ whenever $n \geq N_3$. Hence, for $n \geq N_3$, $\|\tilde{\mathbf{f}}_n(s+h) - \tilde{\mathbf{f}}_n(s)\| \geq |h||\delta - C|h|| > 0$ for all $s \in J_n$ and $h \neq 0$, $|h| \leq \epsilon = \delta/(2C)$ such that $s + h \in J_n$.

Let $K_\epsilon = \{(s, s') \in J \times J : |s' - s| \geq \epsilon\}$. Since K_ϵ is compact and the function $(s, s') \to \|\tilde{\mathbf{f}}(s) - \tilde{\mathbf{f}}(s')\|$ is continuous and positive on K_ϵ, $\delta_1 = \inf_{(s,s') \in K_\epsilon} \|\tilde{\mathbf{f}}(s) - \tilde{\mathbf{f}}(s')\| > 0$. By Lemma 15 (iii)–(iv), $\inf_{(s,s') \in K_\epsilon} \|\tilde{\mathbf{f}}_n(s) - \tilde{\mathbf{f}}_n(s')\| > \delta_1/2 > 0$ for $n \geq N''_4$. The conclusion follows for $n \geq N_4 = \max(N'_4, N''_4)$. $\diamondsuit$

Lemma 17 *Under conditions* $(\mathbf{D1}) - (\mathbf{D4})$,

(i) *there exists* N_6 *such that for all* $n \geq N_6$, $\beta_{n,s}(s') > 0$ *for all* $s' \neq s$;

(ii) *there exists* N_7 *such that for all* $n \geq N_7$ *and a.e.* $s \in J_n$, $\mathbf{f}_n(s)$, $\mathbf{f}'_n(s)$, $\mathbf{f}_n(s')$, $\mathbf{f}'_n(s')$ *and* $\mathbf{f}''_n(s')$ *are linearly independent for all* $s' \neq s$;

(iii) *there exists* N_8 *such that for all* $n \geq N_8$ *and a.e.* $s \in J_n$, $\mathbf{f}_n(s)$, $\mathbf{f}'_n(s)$, $\mathbf{f}_n(s')$, $\mathbf{f}'_n(s')$ *and* $\mathbf{f}_n(s'')$ *are linearly independent for all* $s' \neq s$, $s'' \neq s$, $s' \neq s''$.

Proof. (i) By a Taylor expansion of order 3, it follows from Lemma 15 (iii)-(iv) that $\beta_{n,s}(s + h) = (\tau_{Y_n}(s) + \tau''_{Y_n}(s))h^2/2 + R_n(s, h)$, where $R_n(s, h) \leq C|h|^3$ for some constant $C > 0$. Moreover, by Lemma 16 (i), we have for $n \geq N'_5$ that $\beta_{n,s}(s+h) \geq h^2|\delta/2 - C|h|| > 0$ for all $s \in J_n$ and $h \neq 0$ such that $|h| < \epsilon = \delta/(4C)$.

Since $K_\epsilon = \{(s, s') \in J \times J : |s' - s| \geq \epsilon\}$ is compact, the function $(s, s') \to \beta_s(s')$ is continuous and positive on K_ϵ and $\beta_{n,s}(s')$ converges uniformly to $\beta_s(s')$ for $(s, s') \in J_n \times J_n$ in view of Lemma 15 (iii)–(iv), it follows as in the proof of Lemma 16 (ii) that $\inf_{(s,s') \in K_\epsilon} \beta_{n,s}(s') > 0$ for all $n \geq N''_5$. The conclusion then follows for $n \geq N_5 = \max(N'_5, N''_5)$.

(ii)–(iii) The proof of (ii)-(iii) is similar to the proof of Lemma 16 (ii). First, using a Taylor expansion of sufficient order and Lemma 15, we show that the Gram determinants of both systems are positive for all s', $s' \neq s$ (resp. (s', s''), $s' \neq s$, $s'' \neq s$ and $s' \neq s''$) sufficiently close whenever n is large enough. Second, we take advantage of the continuity and positivity of the Gram determinants of both systems

of sequences for $|s'-s| \geq \epsilon > 0$ (resp. $|s'-s| \geq \epsilon$, $|s''-s| \geq \epsilon$ and $|s'-s''| \geq \epsilon$) and of the uniform convergence of the corresponding Gram determinants as $n \to \infty$. $\diamondsuit$

Lemma 18 *Under conditions* (**D1**), $P\{Z_n \leq a\} \to P\{Z \leq a\}$ *for all* a.

Proof. First, note that

$$|Z - Z_n| \leq \sup_{t\in I} |X(t) - X_n(t)| \leq \frac{1}{\inf_{t\in I} \tau_X(t)} \sup_{t\in I} |U(t) - U_n(t)|.$$

$\kappa_n^2 = \sup_{t\in I} Var(U(t) - U_n(t)) \to 0$ and $d_n^2(t_1, t_2) = E((U(t_1) - U_n(t_1)) - (U(t_2) - U_n(t_2)))^2 \leq C|t_1 - t_2|^2$ for n large enough by Lemma 13. On the other hand, we have the following inequality due to Dmitrovskii (Lifshits, 1986)

$$P\{\sup_{t\in I} |U(t) - U_n(t)| > u\} = 2\exp(-u^2/(2\kappa_n^2))\, q_n(u),$$

where $q_n(u) = 4.1\exp(2^{1/2}6\Psi((\kappa_n/u)^{1/2})(\kappa_n/u)^{1/2}(\Psi((\kappa_n/u)^{1/2}))^{1/2}$ with $\Psi(\epsilon_n) \sim \epsilon_n \log^{1/2}(C_1/\epsilon_n)$ when $\epsilon_n \to 0$ as $n \to \infty$ and $d_n(t_1, t_2) \leq C|t_1 - t_2|$. Therefore, $P\{|Z - Z_n| > u\} \to 0$ as $n \to \infty$ and Z_n converges in probability, hence weakly, to Z. Then $P\{Z_n \leq a\} \to P\{Z \leq a\}$ for all continuity point a. As $P\{Z \leq a\}$ is continuous (Tsirel'son, 1975), $P\{Z_n \leq a\} \to P\{Z \leq a\}$ for all a. $\diamondsuit$

By Lebesgue's convergence theorem and Lemmas 13–17, it follows that $M_n(b) \to M(b)$ for all b and $m_n(b) \to m(b)$ for all $b > 0$. Moreover, $M_n(b) \leq M^*(b)$ for n large enough and $M(b) \leq M^*(b)$ where $M^*(b) = C\varphi(b\theta)$ for some $C > 0$ and $\theta > 0$. Hence, the sequence $\{f_{Z_n}\}$ is weakly compact in $L^1(\mathbb{R})$ (Bourbaki, 1967, pp. 112–113), which means that there exist a function $f \in L^1(\mathbb{R})$ and a subsequence $\{f_{Z_{n'}}\}$ such that $\int_{-\infty}^{\infty} f_{Z_{n'}}(b)h(b)\,db \to \int_{-\infty}^{\infty} f(b)h(b)\,db$ for all $h \in L^\infty(\mathbb{R})$. By taking $h(b) = 1_{(-\infty,a]}(b)$, it results that $P\{Z_{n'} \leq a\} \to \int_{-\infty}^{a} f(b)\,db$ for all a. Therefore, by Lemma 18, Z has a density $f_Z(b) = f(b)$. Moreover, the same result holds for any accumulation point of $\{f_{Z_n}\}$. Finally, $f_{Z_n}(b) \to f_Z(b)$ for a.e. b. By Tsirel'son (1975), $f_Z(b)$ has bounded variation on every compact interval of $\mathbb{R}$. Hence, $f_Z(b)$ has a right and a left limit at each point. If we select a left continuous (say) version of $f_Z(b)$, it follows that $m(b) \leq f_Z(b)$ for all $b > 0$ and $f_Z(b) \leq M(b)$ for all b. $\diamondsuit$

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A Appendix

Let $\mathbf{f}(s) = (f_1(s), \ldots, f_n(s))$, $s \in J = [0, L]$, be a unit speed parameterization of a smooth curve γ embedded in the unit sphere of $\mathbb{R}^n$. Therefore, $\|\mathbf{f}(s)\| = \|\mathbf{f}'(s)\| \equiv 1$, $\langle \mathbf{f}(s), \mathbf{f}''(s)\rangle \equiv -1$, $\|\mathbf{f}''(s)\| > 0$ for all $s \in J$, and $L = |\gamma|$ is the length of γ.

At each point $M = M(s)$ of γ we can define the unit vector tangent to γ $\mathbf{T} = \mathbf{T}(s) = \mathbf{f}'(s)$ and the principal normal vector $\mathbf{N} = \mathbf{N}(s) = \mathbf{f}''(s)/\|\mathbf{f}''(s)\|$. Since $\|\mathbf{T}\| = 1$, $\mathbf{N} = \mathbf{T}'/c$ is orthogonal to $\mathbf{T}$ where $c = c(s) = \|\mathbf{f}''(s)\|$ defines the curvature of γ at the point $M = M(s)$.

For each $s \in J$, if $\mathbf{N}(s) \neq -\mathbf{f}(s)$, there exists only one unit vector $\mathbf{K}{=}\mathbf{K}(s) \in Vect^{\perp}(\mathbf{f}, \mathbf{T})$ such that $\mathbf{K} \in \mathrm{Vect}(\mathbf{f}, \mathbf{N})$ and $\langle \mathbf{N}, \mathbf{K}\rangle = \cos\alpha > 0$. Moreover, $\mathbf{K}(s)$ is C^1 on each interval on which $\langle \mathbf{f}, \mathbf{N}\rangle > -1$, i.e. $c \neq 1$. If $c = 1$ we can define $\mathbf{K}(s) = \mathbf{K}(s_-)$. Let us denote by $\mathbf{K}_1 = \mathbf{K}_1(s), \ldots, \mathbf{K}_{n-2} = \mathbf{K}_{n-2}(s)$, an orthonormal basis of $Vect^{\perp}(\mathbf{f}, \mathbf{T})$ such that $\mathbf{K}_1 \equiv \mathbf{K}$ and the functions $\mathbf{K}_2, \ldots, \mathbf{K}_{n-2}$ are C^1. At each point $M = M(s)$ of γ the moving frame $(M, \mathbf{f}, \mathbf{T}, \mathbf{K}_1, \ldots, \mathbf{K}_{n-2})$ is orthonormal and the matrix of $(\mathbf{f}', \mathbf{T}', \mathbf{K}_1', \ldots, \mathbf{K}_{n-2}')$ with respect to $(\mathbf{f}, \mathbf{T}, \mathbf{K}_1, \ldots, \mathbf{K}_{n-2})$ is antisymmetric:

$$
\begin{array}{cc}
\begin{array}{cccccc} \mathbf{f}' & \mathbf{T}' & \mathbf{K}_1' & \mathbf{K}_2' & \ldots & \mathbf{K}_{n-2}' \end{array} & \\
\begin{pmatrix}
0 & -1 & 0 & 0 & \ldots & 0 \\
1 & 0 & -c_g & 0 & \ldots & 0 \\
0 & c_g & & & & \\
0 & 0 & & & & \\
\vdots & \vdots & & \mathbf{D} & & \\
0 & 0 & & & &
\end{pmatrix}
&
\begin{array}{c} \mathbf{f} \\ \mathbf{T} \\ \mathbf{K}_1 \\ \mathbf{K}_2 \\ \vdots \\ \mathbf{K}_{n-2} \end{array}
\end{array}
$$

where $\mathbf{D}$ is a $(n-2) \times (n-2)$ antisymmetric matrix, $c_g = c_g(s) = \langle \mathbf{K}_1, \mathbf{T}'\rangle = c\langle \mathbf{K}_1, \mathbf{N}\rangle = c\cos\alpha \geq 0$ defines the geodesic curvature of γ at the point $M = M(s)$. By definition of α, we have $\langle \mathbf{f}, \mathbf{N}\rangle^2 = \sin^2\alpha = 1/c^2$ and then $c_g^2 = \|\mathbf{f}''\|^2 - 1$. Note that $c_g = 0$ iff $\cos\alpha = 0$, i.e. $\langle \mathbf{f}, \mathbf{N}\rangle = -1$. Finally, it can be shown that $\mathbf{K} = (\mathbf{f} + \mathbf{f}'')/c_g$.

If $\mathbf{g}(t)$, $t \in I = [0, T]$ is a general parameterization of γ with $\|\mathbf{g}'(t)\| > 0$ for all $t \in I$, the corresponding unit speed parameterization of γ is given by $\mathbf{f}(s) = \mathbf{g}(\lambda^{-1}(s))$, $s \in J$, where $s = \lambda(t) = \int_0^t \|\mathbf{g}'(u)\|\, du$, $t \in I$, is the arc length of γ from 0 to t. We have

$$
\mathbf{T}(s) = \mathbf{f}'(s) = \frac{d\mathbf{f}}{ds} = \frac{d\mathbf{g}}{dt}\frac{dt}{ds} = \frac{\mathbf{g}'(t)}{\lambda'(t)} = \frac{\mathbf{g}'(t)}{\|\mathbf{g}'(t)\|},
$$

$$
c(s)\mathbf{N}(s) = \mathbf{f}''(s) = \frac{d^2\mathbf{f}}{ds^2} = \frac{d}{dt}\left(\frac{\mathbf{g}'}{\|\mathbf{g}'\|}\right)\frac{dt}{ds} = \frac{1}{\|\mathbf{g}'(t)\|^2}\left(\mathbf{g}''(t) - \frac{\langle \mathbf{g}'(t), \mathbf{g}''(t)\rangle}{\|\mathbf{g}'(t)\|^2}\mathbf{g}'(t)\right),
$$

$$
c_g^2(t) = \frac{\|\mathbf{g}'(t)\|^2\|\mathbf{g}''(t)\|^2 - \langle \mathbf{g}'(t), \mathbf{g}''(t)\rangle^2}{\|\mathbf{g}'(t)\|^6} - 1.
$$

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We derive an integral formula for the density of the maximum of smooth Gaussian processes. This expression induces explicit lower and upper bounds which are in general asymptotic to the density. Our constructive approach relies on a geometric representation of Gaussian processes involving a unit speed parameterized curve embedded in the unit sphere.